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HERMITIAN TRANSFORMATIONS OF DEFICIENCY-INDEX (1, 1), JACOBI MATRICES AND UNDETERMINED MOMENT PROBLEMS.*

By Hans Ludwig Hamburger.

CONTENTS.

INTRODUCTION.

0. 1. M. H. Stone has proved [1] that every self-adjoint transformation with simple spectrum can be represented as a self-adjoint Jacobi matrix. In the present paper we answer the question: when can a closed Hermitian prime transformation [2] of deficiency-index $(1, 1)$ be carried into a Jacobi matrix of deficiency index $(1, 1)$? In other words: If a c. H. p. t.[3] H of d. i. $(1, 1)$ is

* Received March 18, 1943.

[1] [8], p. 282, Theorem 7. 13.

[2] See the definition of prime transformations in [2], p. 119, § 7, and [3], p. 79, Definition 2. See the definition of deficiency-index in [4], p. 87, Definition 15; or [8], p. 81, Definition 2. 21; p. 338, Definition 9. 1.

[3] The abbreviations 'c. H. p. t.' and 'd. i' will be used, respectively, for 'closed Hermitian prime transformation' and 'deficiency-index.'

given, we state the necessary and sufficient conditions for the existence of a complete orthonormal set of elements $u_1, u_2, \cdots$ such that

$$(Hu_\nu, u_\mu) = a_{\mu,\nu} = 0 \quad \text{for} \quad |\mu - \nu| \geq 2.$$

If we put $a_{\mu,\mu} = a_\mu$, $a_{\mu,\mu+1} = \bar{a}_{\mu+1,\mu} = b_\mu$, then the $a_{\mu,\nu}$ form the Jacobi matrix

$$(0.11) \qquad J = (a_{\mu,\nu}) = \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \bar{b}_1 & a_2 & b_2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \bar{b}_2 & a_3 & b_3 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \bar{b}_3 & a_4 & b_4 & 0 & \cdot & \cdot & \ddots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The problem dealt with in this paper differs completely, as one might expect, from the case of the self-adjoint transformation and leads to new restrictive conditions for the c. H. p. t. H. The final result is given in Theorem 3, **7. 5**.

0. 2. A further result of Stone [4] implies that the spectrum of every self-adjoint extension of a closed Hermitian transformation J defined by a Jacobi matrix of d. i. (1, 1) is simple, and consists only of an infinite number of isolated points. Moreover, since two different self-adjoint extensions of J have no characteristic value in common,[5] J is also prime; for otherwise [6] there would be at least one characteristic value and one characteristic solution belonging to every self-adjoint extension of J. Therefore, in what follows, we can confine our investigations to c. H. p. t.'s H of d. i. (1, 1) whose self-adjoint extensions have spectra consisting only of an infinite number of isolated points.

In **1** we give a method for constructing all these transformations H, making use of our general results on c. H. p. t.'s of d. i. (1, 1) which we have developed elsewhere.[7] From this we derive in **2-4** a new representation for the resolvent $R_x{}^t$ of any self-adjoint extension $H_x{}^t = H^t - xI$ of H, viz.,

$$(0.21) \qquad R_x{}^t f = S_x f + \frac{p(x) + tu(x)}{q(x) + tv(x)} (f, \phi^*(\bar{x}))\phi^*(x).$$

Here f is any element of the Hilbert space $\mathfrak{H}$ in which H is defined, t is a real variable such that, if t runs from $-\infty$ to $+\infty$, we obtain the resolvents of

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[4] [8], 585, Theorem 10. 41.

[5] This follows from [8], p. 585, Theorem 10. 41.

[6] This readily follows from the definition of a prime transformation.

[7] See **2** and **3**.

all self-adjoint extensions H^t of H. $r(x), q(x), u(x), v(x)$ denote four integral functions with real roots determined by H; $\phi^*(x)$ is a characteristic solution of H^*, the adjoint transformation of H, which is normalized in a special way and belongs to the characteristic value x, so that

$$H^*_x \phi^*(x) = 0.$$

As we have shown elsewhere,[8] this equation has one solution for every x, real or non-real. Finally, S_x is a bounded Hermitian transformation which is independent of t, such that $(S_x f, g)$ is an integral function of x for every pair of elements f, g of $\mathfrak{H}$.

The representation (0.21) readily leads to an expansion in a series of partial fractions

$$R_x{}^t f = \sum_{\alpha=1}^{\infty} \frac{\mu_\alpha(t)}{\lambda_\alpha(t) - x} (f, \phi^*(\lambda_\alpha(t)))\phi^*(\lambda_\alpha(t)),$$

where the $\lambda_\alpha(t)$ denote the roots of $q(x) + tv(x)$, and $-\mu_\alpha(t)$ the residue of $\dfrac{p(x) + tu(x)}{q(x) + tv(x)}$ at the pole $x = \lambda_\alpha(t)$.

In **5** we give a summary of all those properties of Jacobi matrices of d.i. (1, 1) which we use in our further development. This leads, in **6**, **7**, to the statement of the necessary and sufficient conditions for carrying the c. H. p. t. H of d. i. (1, 1) into a Jacobi matrix of d. i. (1, 1).

0.3. Every Jacobi matrix J is associated with a power series, $\sum_{\nu=0}^{\infty} (c_\nu / x^{\nu+1})$, where $c_0 = 1$. This power series, whose radius of convergence can be zero, is characterized by the fact that, by a familiar formal procedure, it can be expanded into an infinite continued fraction[9]

$$(0.31) \qquad \frac{1 \quad |}{| \, a_1 - x} - \frac{| \, b_1 \, |^2 \quad |}{| \, a_2 - x} - \frac{| \, b_2 \, |^2 \quad |}{| \, a_3 - x} - \cdots$$

where the a_ν and b_ν are the elements of the matrix (0.11). If, and only if, the power series $\sum_{\nu=0}^{\infty} (c_\nu / x^{\nu+1})$ can be expanded into a continued fraction (0.31), the sequence of coefficients $\{c_\nu\}$ defines a moment problem;[10] i. e. a monotone-increasing function $\rho(\lambda)$ such that

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[8] [2], p. 120, Theorem 5; [3], p. 94, Theorem 11.

[9] See e. g. [1], I, pp. 247-249. For further references see *loc. cit.*, p. 247, footnote (16).

[10] [1], I, pp. 266-276, especially, Definition III, p. 274. See also [8], p. 606.

$$c_\nu = \int_{-\infty}^{+\infty} \lambda^\nu d\rho(\lambda). \tag{0.32}$$

The function $\rho(\lambda)$, in the case of a linear material distribution, can be interpreted in physical terms as the mass lying in the interval $-\infty, \cdots, \lambda$.

The moment problem is called determined if there is a unique function $\rho(\lambda)$ satisfying (0.32). This coincides with the case in which the Jacobi matrix J associated with the sequence $\{c_\nu\}$ is self-adjoint.[11] The moment problem is called undetermined if (0.32) has more than one solution $\rho(\lambda)$. In this case the associated Jacobi matrix J is of d.i. $(1,1)$, and there are an infinity of solutions $\mu(\lambda)$ which are step-functions,[12] so that we can write (0.32) as an infinite sum

$$c_\nu = \sum_{\alpha=0}^{\infty} \mu_\alpha \lambda_\alpha{}^\nu, \qquad (\mu_\alpha > 0).$$

Since every Jacobi matrix of d.i. $(1,1)$ is associated with a sequence $\{c_\nu\}$ defining an undetermined moment problem, and *vice versa*, the problem of determining all sequences $\{c_\nu\}$ which define an undetermined moment problem is equivalent to that of constructing all c. H. p. t.'s of d.i. $(1,1)$ which can be carried into a Jacobi matrix of d.i. $(1,1)$. Thus the answer to the question put at the beginning of **0.1** leads to the following Theorem concerning the undetermined moment problem, proved in **8**:

THEOREM 4. *We consider the class $\mathfrak{A}$ of all integral functions $q(x)$ of finite order whose roots are all simple and real, which are real for real x, and which fulfill the conditions*

$$\frac{1}{q(x)} = \sum_{\alpha=1}^{\infty} \frac{1}{q'(\lambda_\alpha)(x-\lambda_\alpha)},$$

$$\sum_{\alpha=1}^{\infty} \frac{\lambda_\alpha{}^k}{q'(\lambda_\alpha)} < \infty \qquad (k = 0, 1, 2, \cdots).$$

Then we find all sequences $\{c_\nu\}$ defining an undetermined moment problem by associating with any $q(x)$ of class $\mathfrak{A}$ any sequence $\{\mu_\alpha\}$, $\mu_\alpha > 0$, such that

$$\sum_{\alpha=1}^{\infty} \mu_\alpha = 1, \qquad \sum_{\alpha=1}^{\infty} \mu_\alpha \lambda_\alpha{}^{2k} < \infty, \qquad (k = 1, 2, \cdots \to \infty),$$

$$\sum_{\alpha=1}^{\infty} \frac{1}{\mu_\alpha (q'(\lambda_\alpha))^2} = \infty, \qquad \sum_{\alpha=1}^{\infty} \frac{1}{\mu_\alpha \lambda_\alpha{}^2 (q'(\lambda_\alpha))^2} < \infty.$$

If we put

[11] [8], p. 559, Theorem 10.30.

[12] [1], III, p. 169, Theorem xxix. See also [8], p. 545, Theorem 10.27.

$$c_k = \sum_{\alpha=1}^{\infty} \mu_\alpha \lambda_\alpha^k, \qquad (k = 0, 1, 2, \cdots \to \infty),$$

the sequence $\{c_k\}$ *defines an undetermined moment problem.*

The sequence $\{\mu_\alpha\}$ satisfies all given conditions if, κ_1 and κ_2 being two positive constants, we have

$$\frac{\kappa_1}{(q'(\lambda_\alpha))^2} \leq \mu_\alpha \leq \frac{\kappa_2}{(q'(\lambda_\alpha))^2}.$$

0.4. In the Appendix we derive from a Theorem of Titchmarsh [13] Theorems 5a and 5b, in which we give a sub-class of functions belonging to $\mathfrak{A}$.

Let $q(x)$ *be an integral function of order* ρ, *real for real* x, *having real roots* λ_α. *Let* γ *be any positive constant,* $n_1(r)$ *the number of the positive roots* $\lambda_\alpha \leq r$, *and* $n_2(r)$ *the number of the negative roots for which* $-\lambda_\alpha \leq r$. *Then* $q(x)$ *belongs to* $\mathfrak{A}$ *if*

(i) $n_1(r) \sim \gamma r^\rho$, $n_2(r) = 0$, $0 < \rho < \frac{1}{2}$,

or (ii) $n_1(r) = 0$, $n_2(r) \sim \gamma r^\rho$, $0 < \rho < \frac{1}{2}$,

or (iii) $n_1(r) \sim \gamma r^\rho$, $n_2(r) \sim \gamma r^\rho$, $0 < \rho < 1$.

CHAPTER I.

The resolvents of self-adjoint extensions of a closed Hermitian prime transformation of deficiency-index (1, 1).

1. Construction of closed Hermitian prime transformations of deficiency-index (1, 1).

1.1. Let $\{\lambda_\alpha\}$ $(\alpha = 0, 1, \cdots \to \infty)$ be any infinite set of real numbers which have no finite limit point and which are all different one from another and let χ_α be any complete orthonormal set of elements of the Hilbert space $\mathfrak{H}$. Then

$$H_x{}^0 f = \sum_{\alpha=1}^{\infty} (\lambda_\alpha - x)(f, \chi_\alpha)\chi_\alpha$$

is a self-adjoint transformation, and

$$R_x g = \sum_{\alpha=1}^{\infty} \frac{(g, \chi_\alpha)}{\lambda_\alpha - x} \chi_\alpha$$

is its resolvent. Here g denotes any element of $\mathfrak{H}$ and f an element such that

[13] [9] and [10], pp. 271-272, § 8. 64.

$$\sum_{a=1}^{\infty} \lambda_a^2 \,|(f, \chi_a)|^2 < \infty. \tag{1.11}$$

The set of all elements f satisfying (1.11) is the domain $\mathfrak{D}^0$ of H^0 and the spectrum of H^0 consists only of the set of the λ_a, which are all characteristic values of the first order.

1.2. In order to determine a contraction H of H^0 which is prime and of d.i. (1, 1), we consider an element

$$\Phi(x) = \sum_{a=1}^{\infty} \frac{\Omega_a}{\lambda_a - x} \chi_a \tag{1.21}$$

the Ω_a being complex constants $\neq 0$, such that

$$\sum_{a=1}^{\infty} \frac{|\Omega_a|^2}{\lambda_a^2 + 1} = 1, \qquad \sum_{a=1}^{\infty} |\Omega_a|^2 = \infty. \tag{1.22}$$

Thus, by the general theory developed elsewhere,[14] we obtain the domain $\mathfrak{D}$ of H as the set of all elements $f = R_{-i}g$ where g is any element of $\mathfrak{H}$ satisfying the condition $(g, \Phi(i) = 0$. Hence, for all $f \in \mathfrak{D}$,

$$(H_{-i}f, \Phi(i)) = \sum_{a=1}^{\infty} (f, \chi_a)\bar{\Omega}_a = 0. \tag{1.23}$$

The domain $\mathfrak{D}$ is evidently a subset of $\mathfrak{D}^0$. In this way we can construct all closed Hermitian prime transformations of d.i. (1, 1) whose self-adjoint extensions have a spectrum consisting only of an infinite number of isolated points.

Let H^* be the transformation adjoint to H and $\mathfrak{D}^*$ its domain. Then, in virtue of the general theory *loc. cit.*[14], $\mathfrak{D}^* = \mathfrak{D}^0 \dotplus \mathfrak{M}$, where $\mathfrak{M}$ denotes the linear manifold spanned by $\Phi(i)$. We have further, by (1.23), $H^*_i\Phi(i) = 0$, and, more generally,

$$H^*_x\Phi(x) = 0, \quad \text{for every} \quad x \neq \lambda_a. \tag{1.24}$$

If $\Omega_a = |\Omega_a| e^{iw_a}$ we put $\chi^*_a = e^{iw_a}\chi_a$. Then the χ^*_a form a complete orthonormal set just as well as the χ_a, and $\Phi(x)$ can be represented in the form

$$\Phi(x) = \sum_{a=1}^{\infty} \frac{|\Omega_a|}{\lambda_a - x} \chi^*_a.$$

It is therefore no restriction if we assume henceforth that the Ω_a in (1.21) are all positive numbers.

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[14] [2], pp. 121-125, Theorem 6; [3], p. 84, Theorem 7.

1.3. If $f = \sum_{a=1}^{\infty} f_a \chi_a$, we define a conjugation J by [15]

$$Jf = \sum_{a=1}^{\infty} \bar{f}_a \chi_a.$$

Then H is real with respect to the conjugation J, since $Jf \in \mathfrak{D}$ if $f \in \mathfrak{D}$, because of (1.23) and $\Omega_a > 0$, and since, moreover,

$$JHf = J \sum_{a=1}^{\infty} \lambda_a f_a \chi_a = \sum_{a=1}^{\infty} \lambda_a \bar{f}_a \chi_a = HJf.$$

for every element f of $\mathfrak{D}$. By a theorem of Stone [16] every self-adjoint extension of H is also real with respect to the conjugation J.

2. The function $C_\theta(x)$.

2.1. We write $q(x) = e^{F(x)}\Pi(x)$ where $F(x)$ denotes any integral function, real for real x, and where $\Pi(x)$ denotes Weierstrass's canonical product having the λ_a as roots. We put, furthermore,

$$(2.11) \quad \phi^*(x) = q(x)\Phi(x), \quad \phi(x) = \frac{\phi^*(x)}{|\phi^*(x)|},$$

$$(2.12) \quad \mu(x) = \frac{1}{|\phi^*(x)|^2} = \frac{1}{|q(x)|^2 |\Phi(x)|^2} = \mu(\bar{x}), \quad \frac{\phi(x)}{\sqrt{\mu(x)}} = \phi^*(x),$$

$$(2.13) \quad v(x) = -q(x) \sum_{a=1}^{\infty} \Omega_a^2 \left(\frac{1}{\lambda_a - x} - \frac{1}{\lambda_a} \right),$$

where the series (2.13) converges for every $x \neq \lambda_a$ by (1.22); thus $v(x)$ is also an integral function. Since $\phi^*(x)$ is defined by (2.11) for every x and is everywhere $\neq 0$, we have $\mu(x) > 0$ for every x. (2.13) implies that $v(x)$ as well as $q(x) + tv(x)$ have only simple real roots.

We obtain, moreover, from (2.11) and (1.21)

$$\begin{aligned} (\phi^*(x), \phi^*(y)) &= q(x)q(\bar{y}) (\Phi(x), \Phi(y)) \\ &= q(x)q(\bar{y}) \sum_{a=1}^{\infty} \frac{\Omega_a^2}{(\lambda_a - x)(\lambda_a - \bar{y})} \\ (2.14) \qquad &= \frac{q(x)v(\bar{y}) - v(x)q(\bar{y})}{x - \bar{y}}. \end{aligned}$$

Hence

$$\frac{1}{\mu(x)} = (\phi^*(x), \phi^*(x))$$

[15] [8], p. 357, Definition 9.7.

[16] [8], p. 361, Theorem 9.14, (2).

$$(2.15) \qquad = \begin{cases} \dfrac{q(x)v(\bar{x}) - v(x)q(\bar{x})}{x - \bar{x}} & \text{for non-real } x, \\ q'(x)v(x) - v'(x)q(x) & \text{for real } x, \end{cases}$$

$$(2.16) \qquad \frac{1}{\mu(\lambda_a)} = q'(\lambda_a)v(\lambda_a) = (q'(\lambda_a))^2\Omega_a^2$$

by (2.13),

$$(2.17) \qquad \phi^*(\lambda_a) = -q'(\lambda_a)\Omega_a\chi_a$$

by (2.11) and (1.21),

$$(2.18) \qquad (\phi(x), \phi(y)) = \sqrt{\mu(x)\mu(y)}\,\frac{q(x)v(\bar{y}) - v(x)q(\bar{y})}{x - \bar{y}}, \quad (\phi(x), \phi(x)) =$$

2.2. As we have shown elsewhere,[17] an m-th order matrix $C(x)$ is defined by any self-adjoint extension H^0 of a c. H. p. t. of d. i. (m, m) and is given by the formula [18]

$$2iC(x) = (x + i)\phi(x, i) - (x - i)\phi(x, -i)U^*.$$

Here $\phi(x, i)$ denotes the matrix with the element $(\phi_\mu(x), \phi_\nu(i))$ $(\mu, \nu = 1, 2, \cdots, m)$ and U denotes the unitary matrix which, by von Neumann's theory,[19] determines the extension H^0. If $m = 1$, $C(x)$ and U contain only one element, and we have $U = e^{i\theta}$, $U^* = e^{-i\theta}$. Hence

$$2iC_\theta(x) = (x + i)(\phi(x), \phi(i)) - e^{-i\theta}(x - i)(\phi(x), \phi(-i))$$

and, by (2.12) and (2.18),

$$2iC_\theta(x) = \sqrt{\mu(x)\mu(i)}\,(q(x)(v(-i) - e^{-i\theta}v(i)) - v(x)(q(-i) - e^{-i\theta}q(i))).$$

If we put

$$t = -\frac{q(-i) - e^{-i\theta}q(i)}{v(-i) - e^{-i\theta}v(i)}, \quad e^{-i\theta} = \frac{q(-i) + tv(-i)}{q(i) + tv(i)}, \quad -\infty < t \leq \infty,$$

$$\gamma(t) = \frac{\sqrt{\mu(i)}}{2i}\,(v(-i) - e^{-i\theta}v(i)) = \frac{1}{\sqrt{\mu(i)}\,(q(i) + tv(i))},$$

we obtain finally

$$(2.21) \qquad \begin{aligned} C_\theta(x) &= C(x; t) = \gamma(t)\sqrt{\mu(x)}\,(q(x) + tv(x)) = \frac{\sqrt{\mu(x)}\,(q(x) + tv(x))}{\sqrt{\mu(i)}\,(q(i) + tv(i))}, \\ C(x; \infty) &= \frac{\sqrt{\mu(x)}\,v(x)}{\sqrt{\mu(i)}\,v(i)}. \end{aligned}$$

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[17] [2], p. 117, Theorem 1; [3], p. 68, Theorem 1.

[18] [3], p. 71, formula (13.6). [19] [4], p. 82, Satz 25, and pp. 89-91, § VIII.

3. A preparatory lemma.

3.1. LEMMA 1. *If $H_x{}^t$ denotes that self-adjoint extension of the c. H. p. t. H of d. i. (1, 1) which determines $C(x;t)$, and if $R_x{}^t$ denotes its resolvent, then*

$$R_x{}^t f = S_x f + \frac{p(x,t)}{q(x)+tv(x)}(f, \phi^*(\bar{x}))\phi^*(x) \tag{3.11}$$

for every f of $\mathfrak{H}$. Here $p(x,t)$ is, for every real t, an integral function of x, real for real x, and S_x is a bounded linear transformation with domain $\mathfrak{H}$, such that $(S_x f, g) = (f, S_x g)$ is an integral function of x for every pair of elements f, g of $\mathfrak{H}$.

Proof. If $\mathfrak{R}_x$ is the range of H_x, then, by von Neumann's Theory,[20] $\mathfrak{R}_x$ is the closed linear manifold of all elements h_x of $\mathfrak{H}$ satisfying the condition

$$(h_x, \phi^*(\bar{x})) = 0. \tag{3.12}$$

The equation $H_x\psi = 0$, furthermore, has no solution $\psi \neq 0$ where $\psi \in \mathfrak{D}$, for otherwise the transformation H would not be prime.[21] This implies the existence of a transformation $\tilde{S}_x$ defined in $\mathfrak{R}_x$ and carrying $\mathfrak{R}_x$ into $\mathfrak{D}$, such that

$$\tilde{S}_x H_x f = \tilde{S}_x H_x{}^t f = f, \qquad H_x \tilde{S}_x h_x = H_x{}^t \tilde{S}_x h_x = h_x,$$

if $f \in \mathfrak{D}$ and $h_x \in \mathfrak{R}_x$. From this we obtain

$$\tilde{S}_x h_x = R_x{}^t h_x \tag{3.13}$$

for every h_x of $\mathfrak{R}_x$ and every real t, $t = \infty$ included.

We now show that

$$R_x{}^t f - R_x{}^{t_0} f = \frac{1}{x-\bar{x}}\left(\frac{C(\bar{x};t)}{C(x;t)} - \frac{C(\bar{x};t_0)}{C(x;t_0)}\right)(f, \phi(\bar{x}))\phi(x) \tag{3.14}$$

for $f \in \mathfrak{H}$. We can write f in the form $f = h_x + (f, \phi(\bar{x}))\phi(\bar{x})$, where $h_x \in \mathfrak{R}_x$ satisfies condition (3.12). Since, by (3.13), $(R_x{}^t - R_x{}^{t_0})h_x = 0$ for every element h_x of $\mathfrak{R}_x$, (3.14) holds evidently for $f = h_x$, and has, therefore, to be proved only for $f = \phi(\bar{x})$. By a theorem proved elsewhere [22] we have

$$(y-x)R_x{}^t\phi(y) = \phi(y) - \frac{C(y;t)}{C(x;t)}\phi(x). \tag{3.15}$$

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[20] [4], p. 85, Theorem 28; [8], pp. 143-144, Theorems 4.15 and 4.16.

[21] This readily follows from the definition of a prime transformation given in [2] and [3], *loc. cit.*[2]

[22] [2], p. 117, Theorem 1; [3], p. 68, Theorem 1.

If we put $y = \bar{x}$ we obtain from (3.15)

$$(\bar{x} - x)(R_x{}^t\phi(\bar{x}) - R_x{}^{t_0}\phi(\bar{x})) = -\left(\frac{C(\bar{x};t)}{C(x;t)} - \frac{C(\bar{x};t_0)}{C(x;t_0)}\right)\phi(x),$$

which is the desired result (3.14) for $f = \phi(\bar{x})$.

By (2.21), however, we have

$$\frac{C(\bar{x};t)}{C(x;t)} = \frac{q(\bar{x}) + tv(\bar{x})}{q(x) + tv(x)},$$

and this implies

$$\frac{1}{x - \bar{x}}\left(\frac{C(\bar{x};t)}{C(x;t)} - \frac{C(\bar{x};t_0)}{C(x;t_0)}\right) = \frac{(t - t_0)(q(x)v(\bar{x}) - v(x)q(\bar{x}))}{(x - \bar{x})(q(x) + tv(x))(q(x) + t_0v(x))}$$
$$= \frac{t - t_0}{\mu(x)(q(x) + tv(x))(q(x) + t_0v(x))}$$

because of (2.15). By substituting this expression in (3.14) and by (2.12) we are finally led to

$$R_x{}^tf - R_x{}^{t_0}f = M(x;t,t_0)(f,\phi^*(\bar{x}))\phi^*(x), \tag{3.16}$$

where

$$M(x;t,t_0) = \frac{t - t_0}{(q(x) + tv(x))(q(x) + t_0v(x))}. \tag{3.17}$$

$M(x;t,t_0)$ is a meromorphic function, which, for $t \neq t_0$, has poles of the first order at the roots $x = \lambda_\alpha(t)$ of the integral function $q(x) + tv(x)$ and at the roots $x = \lambda_\alpha(t_0)$ of the function $q(x) + t_0v(x)$.

3.2. We now determine a meromorphic function $m(x;t_0)$ which has only simple poles at $x = \lambda_\alpha(t_0)$ with the same residue as $-M(x;t,t_0)$. $m(x;t_0)$ is determined save for an additional term $s(x)$, $s(x)$ being an integral function of x. Moreover, if we put

$$m(x;t) = m(x;t_0) + M(x;t,t_0), \tag{3.21}$$

then $m(x;t)$ has simple poles only for $x = \lambda_\alpha(t)$ $(t \neq t_0)$ with the same residue as $M(x;t,t_0)$. Hence we can write

$$m(x;t) = \frac{p(x;t)}{q(x) + tv(x)}, \tag{3.22}$$

where $p(x;t)$ is an integral function of x for every t.

We now define a bounded linear transformation $T_x{}^t$ by

$$T_x{}^t f = \frac{p(x;t)}{q(x) + tv(x)} (f, \phi^*(\bar{x}))\phi^*(x), \tag{3.23}$$

which implies by (3.16), (3.21) and (3.22)

$$T_x{}^t f - T_x{}^{t_0} f = R_x{}^t f - R_x{}^{t_0} f. \tag{3.24}$$

If we put

$$S_x{}^t = R_x{}^t - T_x{}^t, \tag{3.25}$$

we obtain from (3.24)

$$S_x{}^t - S_x{}^{t_0} = 0,$$

i. e. the bounded linear transformation $S_x{}^t$ does not depend on t, and therefore we can write S_x instead of $S_x{}^t$. $m(x;t_0)$ being undetermined implies that $S_x f$ is also undetermined to the extent of an additional term of the form

$$S(x)(f, \phi^*(\bar{x})\phi^*(x). \tag{3.26}$$

On the other hand,

$$(S_x f, g) = ((R_x{}^t - T_x{}^t)f, g)$$

is an integral function of x for any $f, g \in \mathfrak{H}$; for (i) $(R_x{}^t f, g)$ is a meromorphic function of x whose poles are the characteristic values of the self-adjoint extension H^t. These are, however, by a theorem proved elsewhere [23] the zeros of $C(x;t)$, i. e. $x = \lambda_a(t)$ because of (2.21); (ii) if $M_a(t)$ is the residue of $M(x;t,t_0)$ at the pole $x = \lambda_a(t)$, then, by (3.16), the residue of $(R_x{}^t f, g)$ equals $M_a(t)(f, \phi^*(\lambda_a(t))(\phi^*(\lambda_a(t)), g)$; (iii) $(f, \phi^*(\bar{x}))$ and $(\phi^*(x), g)$, being integral functions by (2.11), $(T_x{}^t f, g)$ is a meromorphic function by (3.23) with simple poles at $x = \lambda_a(t)$; (iiii) The residues of $(R_x{}^t f, g)$ and $(T_x{}^t f, g)$ at the poles $x = \lambda_a(t)$ coincide, because $m(x;t)$ and $M(x;t,t_0)$ have the same residues at $x = \lambda_a(t)$.

We obtain furthermore from (3.23) $(T_x{}^t f, g) = (f, T_{\bar{x}}{}^t g)$ which, by (3.25), implies $(S_x f, g) = (f, S_{\bar{x}} g)$. (3.23) and (3.25), finally, lead to (3.11), which completes the proof.

4. Analytical representation of $R_x{}^t$.

4.1. By substituting in (3.21) the expressions for $M(x;t,t_0)$, $m(x;t)$ and $m(x;t_0)$ given by (3.17) and (3.22), respectively, we are led to

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[23] [2], p. 120; [3], pp. 88-89, Theorem 9.

$$(4.11)\qquad (q(x)+t_0v(x))p(x;t)-(q(x)+tv(x))p(x;t_0)=t-t_0.$$

If we put

$$t_0=0,\qquad p(x;0)=p(x),$$

we obtain from (4.11)

$$q(x)p(x;t)-(q(x)+tv(x))p(x)=t,$$
$$p(x;t)=p(x)+t\,\frac{v(x)p(x)+1}{q(x)}.$$

Since $p(x;t)$ is an integral function of x for every real t,

$$u(x)=\frac{v(x)p(x)+1}{q(x)}$$

is also an integral function; moreover, we have

$$(4.12)\qquad p(x;t)=p(x)+tu(x),$$
$$(4.13)\qquad u(x)q(x)-p(x)v(x)=1.$$

Thus, by (4.12), we obtain from (3.11) the final expression for R_x^t, namely,

$$(4.14)\qquad R_x^tf=S_xf+\frac{p(x)+tu(x)}{q(x)+tv(x)}\,(f,\phi^*(\bar{x}))\phi^*(x).$$

S_xf being determined save for a term of the form (3.26), $p(x)$ and $u(x)$ are determined save for additional terms $-s(x)q(x)$ and $-s(x)v(x)$ respectively.

4.2. The general theory of the resolvents [24] yields for R_x^t also a representation by partial fractions, which we derive readily from (4.14). By (4.14) we notice that (R_x^tg,f) has its poles at the roots $x=\lambda_a(t)$ of $q(x)+tv(x)$ with the residue

$$\frac{p(\lambda_a(t))+tu(\lambda_a(t))}{q'(\lambda_a(t))+tv'(\lambda_a(t))}\,(g,\phi^*(\lambda_a(t)))(\phi^*(\lambda_a(t)),f).$$

Since $t=-\dfrac{q(\lambda_a(t))}{v(\lambda_a(t))}$ we have, by (2.15) and (4.13),

$$\frac{p(\lambda_a(t))+tu(\lambda_a(t))}{q'(\lambda_a(t))+tv'(\lambda_a(t))}=\frac{p(\lambda_a(t))v(\lambda_a(t))-q(\lambda_a(t))u(\lambda_a(t))}{q'(\lambda_a(t))v(\lambda_a(t))-q(\lambda_a(t))v'(\lambda_a(t))}=-\mu(\lambda_a(t))$$

[24] [4], pp. 91-96, § IX; [8], p. 176, Theorem 5.7.

From this we obtain

$$R_x{}^t f = \sum_{a=1}^{\infty} \frac{\mu(\lambda_a(t))}{\lambda_a(t) - x} (f, \phi^*(\lambda_a(t)))\phi^*(\lambda_a(t)), \tag{4.21}$$

$$H_x{}^t g = \sum_{a=1}^{\infty} (\lambda_a(t) - x)\mu(\lambda_a(t))(g, \phi^*(\lambda_a(t)))\phi^*(\lambda_a(t)), \tag{4.22}$$

which implies that the spectrum of the self-adjoint extension H^t consists only of the characteristic values $\lambda_a(t)$, to which the characteristic solutions $\phi^*(\lambda_a(t))$ correspond. Therefore, for every fixed real value of t as well as for $t = \infty$, the $\phi(\lambda_a(t))$ form a complete orthogonal set by (2.11).

We summarize the result of this Chapter in

THEOREM 1. *Let H be a c.H.p.t. of d.i. $(1,1)$, such that the spectrum of one of its self-adjoint extensions consists only of an infinite number of isolated points. Then there exist four integral functions $p(x)$, $q(x)$, $u(x)$, $v(x)$ which satisfy* (4.13) *and which determine the resolvents $R_x{}^t$ of all self-adjoint extensions $H_x{}^t$ by* (4.14) *and* (4.21).

4.3. *Remark.* Ae we have noticed in **2.1**, the function $q(x)$ is undetermined to the extent of a factor $e^{F(x)}$. If, therefore, we replace $q(x)$ by

$$\tilde{q}(x) = e^{F(x)}q(x),$$

where $F(x)$ denotes an integral function, real for real x, and set

$$\begin{aligned} \tilde{v}(x) &= e^{F(x)}v(x), & & \\ \tilde{p}(x) &= e^{-F(x)}p(x), & \tilde{u}(x) &= e^{-F(x)}u(x), \\ \tilde{\phi}^*(x) &= e^{F(x)}\phi^*(x), & \tilde{\mu}(x) &= e^{-F(x)-F(\bar{x})}\mu(x), \end{aligned}$$

we notice readily that the representations (4.14) and (4.21) for $R_x{}^t$ remain unchanged if we replace p, u, q, v, ϕ^*, μ by $\tilde{p}, \tilde{u}, \tilde{q}, \tilde{v}, \tilde{\phi}^*, \tilde{\mu}$, respectively.

CHAPTER II.

The undetermined moment problem.

5. The Jacobi matrix of deficiency index (1, 1).

5.1. In this Chapter we give a development which leads to the necessary and sufficient conditions that a c.H.p.t. H of d.i. $(1,1)$ can be carried into a Jacobi matrix of d.i. $(1,1)$, considering that the determination of all Jacobi matrices of d.i. $(1,1)$ results in the construction of all undetermined moment

problems. We first give a summary of the main properties of Jacobi matrices of d. i. (1, 1).

Let $J = (a_{\mu,\nu})$ be an infinite Jacobi matrix of the form (0.11); then J determines a linear transformation H^* in the concrete Hilbert space $\mathfrak{H}_0$ of all vectors

$$g = (g_1, g_2, \cdots) = \sum_{\nu=1}^{\infty} g_\nu u_\nu$$

with $\sum_{\nu=1}^{\infty} | g_\nu |^2 < \infty$. Here the u_ν denote a complete orthonormal set in $\mathfrak{H}_0$ forming the coördinate-system of the vector-space $\mathfrak{H}_0$. We put

$$(5.11) \qquad (H^* u_\nu, u_\mu) = a_{\mu,\nu}, \qquad H^* g = \sum_{\mu=1}^{\infty} \left(\sum_{\nu=1}^{\infty} a_{\mu,\nu} g_\nu \right) u_\mu,$$

which holds for every vector of a domain $\mathfrak{D}^*$ defined by the condition that

$$\sum_{\mu=1}^{\infty} \left| \sum_{\nu=1}^{\infty} a_{\mu,\nu} g_\nu \right|^2 < \infty.$$

Let f be a vector, such that

$$(5.12) \qquad (H^* f, g) = (f, H^* g)$$

for every g of $\mathfrak{D}^*$, and let the subset $\mathfrak{D}$ of $\mathfrak{D}^*$ be the domain of all vectors f satisfying (5.12). If H is the transformation with domain $\mathfrak{D}$ which coincides with H^* in $\mathfrak{D}$, so that $H \subset H^*$, then H is a c. H. p. t., since $(Hf, f') = (f, Hf')$, if f and $f' \in \mathfrak{D}$.

It is no restriction to assume henceforth that $b_\nu > 0$; for if $b_\nu = | b_\nu | e^{i\beta_\nu}$, then, by a suitable transformation of the coördinate system, $u_\nu = e^{i\theta_\nu} u'_\nu$, where $\theta_1 = 0$, $\theta_\nu = \sum_{\kappa=1}^{\nu-1} \beta_\kappa$ $(\nu \geq 2)$, we can always reduce any matrix J to a Jacobi matrix J' with elements $a'_\nu = a_\nu$, $b'_\nu = | b_\nu |$.

The supposition that J is of d. i. (1, 1) implies that there are elements g of $\mathfrak{D}^*$ which are not elements of $\mathfrak{D}$. Moreover, by a remark made at the beginning of **0.2**, H is a prime transformation.

5.2. We now introduce [25] an infinite sequence of polynomials $G_n(x)$ of degree $n - 1$ determined by the recursion formulae

$$(5.21) \qquad \begin{cases} G_1(x) = 1, \quad b_1 G_2(x) + a_1 - x = 0 \\ b_{n-1} G_n(x) + (a_{n-1} - x) G_{n-1}(x) + b_{n-2} G_{n-2}(x) = 0 \quad (n \geq 3). \end{cases}$$

[25] [8], p. 531.

It then follows [26] from J being of d.i. (1, 1) that $\sum_{n=1}^{\infty} | G_n(x)|^2 < \infty$ for any real or complex x. If we put

$$(5.22) \qquad (1/\mu(x)) = \sum_{n=1}^{\infty} | G_n(x)|^2,$$

$$(5.23) \qquad \phi^*(x) = \sum_{\nu=1}^{\infty} G_\nu(x) u_\nu,$$

$$(5.24) \qquad \phi(x) = \sqrt{\mu(x)}\, \phi^*(x),$$

the equations (5.21), by (5.11), can be written as

$$(5.25) \qquad H^*_x \phi^*(x) = 0.$$

We furthermore determine a second sequence of polynomials, $H_n(x)$, of degree $n-2$ by the recursion formulae [27]

$$(5.26) \qquad \begin{cases} H_1(x) = 0, \quad b_1 H_2(x) = -1 \\ b_{n-1} H_n(x) + (a_{n-1} - x) H_{n-1}(x) + b_{n-2} H_{n-2}(x) = 0 \quad (n \geq 3). \end{cases}$$

Then, as a consequence of our supposition concerning J, we have $\sum_{n=1}^{\infty} | H_n(x)|^2 < \infty$. If we put

$$(5.27) \qquad \psi(x) = \sum_{\nu=1}^{\infty} H_\nu(x) u_\nu,$$

we are led by (5.11) and (5.26) to

$$(5.28) \qquad H^*_x \psi(x) = -u_1.$$

Moreover, we obtain the following equations [28]

$$(5.29) \qquad (x-y) \sum_{\nu=1}^{n} G_\nu(x) G_\nu(y) = b_n(G_{n+1}(x) G_n(y) - G_n(x) G_{n+1}(y)),$$

$$(5.210) \qquad 1 + (x-y) \sum_{\nu=1}^{n} G_\nu(x) H_\nu(y) = b_n(G_{n+1}(x) H_n(y) - G_n(x) H_{n+1}(y),$$

$$(5.211) \qquad 1 = b_n(G_{n+1}(x) H_n(x) - H_{n+1}(x) G_n(x)).$$

We readily verify that the vector u_1, as well as the vectors $u^*_{k+1} = H^k u_1$, for every integer k, belong to $\mathfrak{D}$, since only a finite number of coördinates of

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[26] [8], p. 546, Theorem 10.27. See also [1], I, p. 301, Theorem XVII; II, p. 136, Hilfsatz 7; [5], pp. 21-23.

[27] [8], p. 539, Theorem 10.25 (1).

[28] [1], I, p. 256, formulae (39), (40); [8], p. 536, Theorem 10.24; p. 540, Theorem 10.25 (2).

these vectors are $\neq 0$. This implies by (5.23) and (5.25), for $k = 0, 1, 2, \cdots$, that

$$(5.212)\qquad \begin{aligned}(\phi^*(x), H^k u_1) &= (H^{*k}\phi^*(x), u_1)\\ &= x^k(\phi^*(x), u_1) = x^k G_1(x) = x^k.\end{aligned}$$

The coördinate vectors u_ν can now be obtained by orthogonalizing the vectors u^*_{k+1} (by E. Schmidt's familiar procedure).

Finally we notice that $\dfrac{H_{n+1}(x)}{G_{n+1}(x)}$ can be expressed as a continued fraction [29] by

$$(5.213)\quad \frac{H_{n+1}(x)}{G_{n+1}(x)} = \frac{1\;|}{|\,a_1 - x} - \frac{b_1^2\;|}{|\,a_2 - x} - \cdots - \frac{b^2_{n-1}\;|}{|\,a_{n-1} - x}, \quad (n \geqq 1),$$

which is an approximant of (0.31).

5.3. We now put [30]

$$\begin{aligned}P_n(x) &= b_n(H_{n+1}(x)G_n(0) - H_n(x)G_{n+1}(0)),\\ Q_n(x) &= b_n(G_{n+1}(x)G_n(0) - G_n(x)G_{n+1}(0)),\\ U_n(x) &= b_n(H_{n+1}(x)H_n(0) - H_n(x)H_{n+1}(0)),\\ V_n(x) &= b_n(G_{n+1}(x)H_n(0) - G_n(x)H_{n+1}(0)),\end{aligned}$$

so that by (5.211)

$$(5.31)\qquad \begin{cases} P_n(0) = -1, & Q_n(0) = 0,\\ U_n(0) = 0, & V_n(0) = 1.\end{cases}$$

Then we obtain readily by (5.29), (5.210) and (5.211)

$$(5.32)\quad \begin{aligned}Q_n(x)V_n(y) - V_n(x)Q_n(y) &= b_n(G_{n+1}(x)G_n(y) - G_n(x)G_{n+1}(y))\\ &= (x-y)\sum_{\nu=1}^{n} G_\nu(x)G_\nu(y),\end{aligned}$$

$$(5.33)\quad Q_n(x)U_n(y) - V_n(x)P_n(y) = 1 + (x-y)\sum_{\nu=1}^{n} G_\nu(x)H_\nu(y),$$

$$(5.34)\quad Q_n(x)U_n(x) - V_n(x)P_n(x) = 1.$$

Moreover, in the case where J is of d. i. (1, 1), we have four integral functions $p(x)$, $q(x)$, $u(x)$, $v(x)$ such that [31]

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[29] [8], p. 540, Theorem 10.25 (5).

[30] [8], pp. 543-544, Theorem 10.26. See also [1], I, p. 254, formula (27); [1], II, p. 121, formula (4). Here the four polynomials are denoted by P_{2n-1}, Q_{2n-1}, G_n, H_n respectively.

[31] The existence of the four functions p, q, u, v has been proved first in [1], where they are denoted by r, s, g, h respectively. See [1], I, p. 310, Theorem xix; [1], II, p. 139, Theorem xx, p. 141, Theorem xxi; [5], pp. 30-31; [8], pp. 565-567, Theorem 10.33.

$$(5.35)\quad \begin{cases} \lim\limits_{n\to\infty} P_n(x) = p(x), & \lim\limits_{n\to\infty} Q_n(x) = q(x), \\ \lim\limits_{n\to\infty} U_n(x) = u(x), & \lim\limits_{n\to\infty} V_n(x) = v(x) \end{cases}$$

for any complex x. This implies, however, by (5.31), (5.32), (5.33) and (5.34), respectively,

$$(5.36)\quad \begin{cases} p(0) = -1, & q(0) = 0, \\ u(0) = 0, & v(0) = 1, \end{cases}$$

$$(5.37)\quad \begin{aligned} q(x)v(y) - v(x)q(y) &= (x-y)\sum_{\nu=1}^{\infty} G_\nu(x)G_\nu(y) \\ &= (x-y)(\phi^*(x), \phi^*(\bar{y})), \end{aligned}$$

$$(5.38)\quad \frac{1}{\mu(x)} = (\phi^*(x), \phi^*(x)) = \begin{cases} \dfrac{q(x)v(\bar{x}) - v(x)q(\bar{x})}{x-\bar{x}} & \text{for non-real } x, \\ q'(x)v(x) - v'(x)q(x) & \text{for real } x, \end{cases}$$

$$(5.39)\quad q(x)u(x) - v(x)p(x) = 1,$$

$$(5.310)\quad \begin{aligned} q(x)u(y) - v(x)p(y) &= 1 + (x-y)\sum_{\nu=1}^{\infty} G_\nu(x)H_\nu(y) \\ &= 1 + (x-y)(\phi^*(x), \psi(\bar{y})), \end{aligned}$$

because of (5.23) and (5.27). All four functions $p(x)$, $q(x)$, $u(x)$, $v(x)$, moreover,[32] have only simple real roots and are of order 1.

5.4. We now consider the meromorphic function of x

$$(5.41)\quad m(x;t) = \frac{q(x) + tv(x)}{p(x) + tu(x)}$$

for every real t $(-\infty < t \leq \infty)$. This function has simple poles at the roots $x = \lambda_\alpha(t)$ of $q(x) + tv(x)$, which are all real. The residue for $x = \lambda_\alpha(t)$ is

$$\frac{p(\lambda_\alpha(t)) + tu(\lambda_\alpha(t))}{q'(\lambda_\alpha(t)) + tv'(\lambda_\alpha(t))} = \frac{-1}{v(\lambda_\alpha(t))q'(\lambda_\alpha(t)) - q(\lambda_\alpha(t))v'(\lambda_\alpha(t))} = -\mu(\lambda_\alpha(t)) < 0,$$

because of (5.38), (5.39) and $t = -\dfrac{q(\lambda_\alpha(t))}{v(\lambda_\alpha(t))}$. Moreover, $m(x;t)$ can be expanded in a series of partial fractions[33]

$$(5.42)\quad m(x;t) = \sum_{\alpha=1}^{\infty} \frac{\mu(\lambda_\alpha(t))}{\lambda_\alpha(t) - x},$$

and is represented in the sectors

[32] [1], I, pp. 315-318. A more exact result is given by M. Riesz in [6], III, pp. 37-44, where he proves that $M(r) < \gamma e^{\epsilon r}$ for every $\epsilon > 0$. Here $M(r)$ denotes the maximum modulus of any of the four functions p, q, u, v, and γ denotes a constant.

[33] [1], III, p. 169, Theorem xxix; [8], p. 569.

$$0 < \delta \leq \arg x \leq \pi - \delta \text{ and } \pi + \delta \leq \arg x \leq 2\pi - \delta$$

by the asymptotic power series [34]

$$(5.43) \qquad -\sum_{\nu=0}^{\infty} (c_\nu / x^{\nu+1}), \qquad c_\nu = \sum_{\alpha=1}^{\infty} \mu(\lambda_\alpha(t))(\lambda_\alpha(t))^\nu, \qquad c_0 = 1,$$

where the c_ν do not depend on t. The power series (5.43) can be expanded, by a familiar formal procedure, in an infinite continued fraction (0.31) whose coefficients a_ν, b_ν coincide with the elements of the Jacobi matrix (0.11).

The c_ν appear in (5.43) as the moments of ν-th order of a distribution of masses which, for any real value of t, concentrates the mass $\mu(\lambda_\alpha(t))$ in the point $\lambda_\alpha(t)$. Since we obtain from (5.43) different distributions of masses for different values of t, the moment problem defined by the sequence (5.43) of coefficients $\{c_\nu\}$ is undetermined. Conversely every undetermined moment problem leads to a Jacobi matrix of d.i. (1, 1) by expanding the power series (5.43) in the associated infinite continued fraction (0.31).

The undetermined moment problem defined by the sequence $\{c_\nu\}$ has solutions other than the solutions (5.42). However, the most general solution of this problem, which has been given first by R. Nevanlinna,[35] can also be expressed by the $p(x)$, $q(x)$, $u(x)$, $v(x)$, so that every solution of a given undetermined moment problem is known, when these four functions are determined. The solutions (5.43), moreover, have the special property [36] that the mass of every other solution

$$c_\nu = \int_{-\infty}^{+\infty} \lambda^\nu d\rho(\lambda)$$

which is concentrated in the point $\lambda = \lambda_\alpha(t)$, is $< \mu(\lambda_\alpha(t))$. Therefore we call the solutions (5.43) maximal distributions of masses.

5.5. Finally we obtain the resolvents of all self-adjoint extensions of H defined by the Jacobi matrix J of d.i. (1, 1) by putting [37]

$$(5.51) \qquad R_x{}^t g = \sum_{\alpha=1}^{\infty} \frac{\mu(\lambda_\alpha(t))}{\lambda_\alpha(t) - x} (g, \phi^*(\lambda_\alpha(t)))\phi^*(\lambda_\alpha(t)).$$

Thus we notice that every solution of the moment problem which consists of a maximal distribution of masses is associated with the resolvent of one of the

[34] [1], I, p. 268, Theorem ix, p. 287; Theorem xiii; [5], pp. 45-49; [8], p. 546, Theorem 10.27 (3).

[35] [5], pp. 33-34, p. 52. See also [8], p. 577, Theorem 10.38.

[36] [1], III, p. 169, Theorem xxix.

[37] [8], pp. 581-582, Theorem 10.39 (5).

self-adjoint extensions of H. If $H_x{}^t = (R_x{}^t)^{-1}$, then the $\lambda_a(t)$ and $\phi^*(\lambda_a(t))$ furnish all characteristic values and characteristic solutions of H^t, respectively.

5.6. If we compare the functions $p(x)$, $q(x)$, $u(x)$, $v(x)$ of Theorem 1, **4.2**, with the functions (5.35), we notice that the equations (2.14) and (4.13) coincide respectively with (5.37) and (5.39). The functions of Theorem 1, however, do not necessarily satisfy the equations (5.36). Therefore we introduce the functions

$$(5.61)\quad \begin{cases} \hat{p}(x) = v(0)p(x) - q(0)u(x), & \hat{u}(x) = u(0)p(x) - p(0)u(x), \\ \hat{q}(x) = v(0)q(x) - q(0)v(x), & \hat{v}(x) = u(0)q(x) - p(0)v(x), \end{cases}$$

which satisfy (5.36) by (4.13). If we try to express the function (3.22) in terms of $\hat{p}(x)$, $\hat{q}(x)$, $\hat{u}(x)$, $\hat{v}(x)$, we obtain

$$m(x;t) = \frac{p(x) + tu(x)}{q(x) + tv(x)} = \frac{\hat{p}(x) + \hat{t}\hat{u}(x)}{\hat{q}(x) + \hat{t}\hat{v}(x)},$$

where

$$\hat{t} = -\frac{q(0) + tp(0)}{v(0) + tu(0)}.$$

Thus the functions $\hat{p}(x)$, $\hat{q}(x)$, $\hat{u}(x)$, $\hat{v}(x)$ correspond exactly to the functions (5.35).

On the other hand, we readily verify that, by (4.13) and (5.61),

$$(5.62)\quad \begin{cases} \hat{q}(x)\hat{v}(y) - \hat{v}(x)\hat{q}(y) = q(x)v(y) - v(x)q(y), \\ \hat{q}'(x)\hat{v}(x) - \hat{v}'(x)\hat{q}(x) = q'(x)v(x) - v'(x)q(x), \\ \hat{q}(x)\hat{u}(y) - \hat{v}(x)\hat{p}(y) = q(x)u(y) - v(x)p(y). \end{cases}$$

From this it follows, by (2.15) and (5.42), that the representation (4.21) for $R_x{}^t g$ has already the desired form (5.51), if we interpret the $\lambda_a(t)$ in (4.21) as the roots of $\hat{q}(x) + \hat{t}\hat{v}(x)$.

5.7. In order to carry the c. H. p. t. H of d. i. (1, 1), defined in **1.2** and **1.3**, into a Jacobi matrix, we first try to determine an element $\psi(x)$ satisfying the equations (5.310) considering that, as we have seen in **5.3**, every Jacobi matrix implies the existence of such an element. By (5.62) we can substitute the functions of Theorem 1 in (5.310) for $q(x)$, $v(x)$, $p(y)$, $u(y)$.

Afterwards we have to construct a complete orthonormal set u_ν which carries the transformation H into a Jacobi matrix by

$$(Hu_\nu, u_\mu) = (a_{\mu,\nu}) = J,$$

and which furnishes the coördinate system of the vector-space $\mathfrak{H}_0$. We shall be led to u_1 by formula (5.28) and to the other elements $u_2, u_3, \cdots$ by an argument referring to (5.212), which was given by M. H. Stone.[38]

6. The element $\psi(x)$.

6.1. Henceforth we use the abbreviations

$$\phi_a = \phi(\lambda_a), \quad \phi^*_a = \phi^*(\lambda_a), \quad \mu_a = \mu(\lambda_a),$$
$$\phi_a(t) = \phi(\lambda_a(t)), \quad \phi^*_a(t) = \phi^*(\lambda_a(t)), \quad \mu_a(t) = \mu(\lambda_a(t)).$$

THEOREM 2. *Let H be a c.H.p.t. of d.i. (1,1) which fulfills the supposition of Theorem 1 and let $p(x)$, $q(x)$, $u(x)$, $v(x)$ be the functions defined in Theorem 1. Then there is an element $\psi(x)$ of $\mathfrak{H}$ satisfying the equation*

$$(\psi(x), \phi^*(\bar{y})) = \frac{u(x)q(y) - p(x)v(y) - 1}{y - x} \tag{6.11}$$

if, and only if, $1/q(x)$ has an expansion in a series of partial fractions, such that

$$\frac{1}{q(x)} - \frac{1}{q(y)} = (y - x) \sum_{a=1}^{\infty} \frac{1}{q'(\lambda_a)(\lambda_a - x)(\lambda_a - y)}, \tag{6.12}$$

and if

$$\sum_{a=1}^{\infty} \frac{\mu_a}{\lambda_a^2} < \infty. \tag{6.13}$$

Moreover, $\psi(x)$ has the form

$$\psi(x) = p(x)\Phi(x) - \sum_{a=1}^{\infty} \frac{\mu_a}{\lambda_a - x} \phi^*_a. \tag{6.14}$$

Proof. In order to show first that $\psi(x)$, if it exists, is necessarily given by (6.14), we put

$$\psi(x) = \sum_{a=1}^{\infty} \psi_a(x) \phi^*_a$$

considering that, by a remark at the end of **4.2**, the ϕ_a form a complete orthonormal set. We then obtain from (6.11), by (2.12) and (2.16),

$$(\psi(x), \phi^*_a) = \frac{\psi_a(x)}{\mu_a} = -\frac{1 + p(x)v(\lambda_a)}{\lambda_a - x},$$

$$\psi_a(x) = -\frac{\mu_a + p(x)\mu_a v(\lambda_a)}{\lambda_a - x} = -\frac{\mu_a + \dfrac{p(x)}{q'(\lambda_a)}}{\lambda_a - x},$$

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[38] [8], pp. 285-286.

$$\psi(x) = -\sum_{a=1}^{\infty}\left(\mu_a + \frac{p(x)}{q'(\lambda_a)}\right)\frac{\phi^*_a}{\lambda_a - x}. \tag{6.15}$$

By (1.21) and (2.17), on the other hand, we have

$$\Phi(x) = -\sum_{a=1}^{\infty}\frac{1}{q'(\lambda_a)(\lambda_a - x)}\phi^*_a,$$

which shows that (6.15) and (6.14) coincide. This, however, implies condition (6.13).

We now, conversely, form $(\psi(x), \phi^*(\bar{y}))$ by using (6.14), and obtain from (2.11), (2.14), (2.16) and (4.13)

$$(\psi(x), \phi^*(\bar{y})) = \frac{p(x)}{q(x)}\,\frac{q(x)v(y) - v(x)q(y)}{x - y} + \sum_{a=1}^{\infty}\frac{\mu_a}{\lambda_a - x}\,\frac{v(\lambda_a)q(y)}{\lambda_a - y}$$
$$= \frac{p(x)v(y) - u(x)q(y)}{x - y} + q(y)\left(\frac{1}{(x-y)q(x)} + \sum_{a=1}^{\infty}\frac{1}{q'(\lambda_a)(\lambda_a - x)(\lambda_a - y)}\right),$$

which coincides with (6.11) if, and only if, (6.12) is satisfied. This completes the proof.

7. The coördinate system of the Jacobi matrix.

7.1. According to a remark at the end of **5.7**, we now have to form $H^*_x\psi(x)$, which, by (5.28), furnishes the coördinate-vector u_1 of the vector space $\mathfrak{H}_0$ carrying H into a Jacobi matrix J. Therefore we first have to find the conditions that $\psi(x) \in \mathfrak{D}^*$, where $\mathfrak{D}^*$ is the domain of H^*.

Let H^0, as in **1**, denote the self-adjoint extension H^t for $t = 0$, and $\mathfrak{D}^0$ the domain of H. By a theorem of von Neumann [39] every element f^* of $\mathfrak{D}^*$ can be represented in the form

$$f^* = f + a\Phi(x) + b\Phi(\bar{x}),$$

where f is an element of $\mathfrak{D}$, x is any non-real number and a, b are constants. If we write this in the form

$$f^* = f - b(\Phi(x) - \Phi(\bar{x})) + (a + b)\Phi(x),$$

we notice readily that

$$\Phi(x) - \Phi(\bar{x}) = (x - \bar{x})\sum_{a=1}^{\infty}\frac{\Omega_a}{(\lambda_a - x)(\lambda_a - \bar{x})}\chi_a$$

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[39] [4], p. 85, Theorem 29; [8], p. 341, Theorem 9.4.

is an element of $\mathfrak{D}^0$, because of

$$| H^0(\Phi(x) - \Phi(\bar{x}))|^2 = | x - \bar{x} |^2 \sum_{a=1}^{\infty} \frac{\lambda_a{}^2\Omega_a{}^2}{| \lambda_a - x |^4} < \infty$$

by (1.21). This implies, however, that every element f^* of $\mathfrak{D}^*$ can also be represented in the form

$$f^* = f^0 + c\Phi(x),$$

where $f^0 \epsilon \mathfrak{D}^0$. Hence $\psi(x) \epsilon \mathfrak{D}^*$ if

$$\psi(x) - p(x)\bar{\Phi}(x) \epsilon \mathfrak{D}^0.$$

By (6.14), (2.17) and (2.12) we obtain

$$\psi(x) - p(x)\bar{\Phi}(x) = \sum_{a=1}^{\infty} \frac{\mu_a q'(\lambda_a)}{\lambda_a - x} \Omega_a \chi_a,$$

and this, by (1.11), is an element of $\mathfrak{D}^0$ if, and only if, (see also (2.16))

$$\sum_{a=1}^{\infty} | \lambda_a |^2 \left| \frac{\mu_a q'(\lambda_a)\Omega_a}{\lambda_a - x} \right|^2 = \sum_{a=1}^{\infty} \left| \frac{\lambda_a}{\lambda_a - x} \right|^2 \mu_a < \infty. \tag{7.11}$$

Now, by (5.43), the condition

$$\sum_{a=1}^{\infty} \mu_a = 1 \tag{7.12}$$

is necessary to reduce H to a Jacobi matrix. Hence $\psi(x) \epsilon \mathfrak{D}^*$, if (7.12) is satisfied, since (7.12) implies (7.11).

(6.14) leads, by (1.24) and (2.12), to

$$H^*{}_x\psi(x) = - H^0{}_x \sum_{a=1}^{\infty} \frac{\mu_a}{\lambda_a - x} \phi^*{}_a = - \sum_{a=1}^{\infty} \mu_a \phi^*{}_a = - \sum_{a=1}^{\infty} \sqrt{\mu_a} \phi_a,$$

and so we obtain from (5.28)

$$u_1 = \sum_{a=1}^{\infty} \sqrt{\mu_a}\, \phi_a, \qquad | u_1 |^2 = \sum_{a=1}^{\infty} \mu_a = 1, \tag{7.13}$$

Finally, we infer from (5.41) and (5.42) that

$$\frac{p(x)}{q(x)} = \sum_{a=1}^{\infty} \frac{\mu_a}{\lambda_a - x}.$$

This equation determines the additional term $-s(x)q(x)$ of $p(x)$, which until now has been undetermined.

7.2. We now have to determine the conditions that for every integer $k \geq 0$, the element $u^*_{k+1} = H^k u_1$ exists, and belongs to $\mathfrak{D}$, which conditions are necessary for reducing H to a Jacobi matrix, as we have shown at the end of **5.2.**

If $u^*_{k+1} \in \mathfrak{D}$, we have $u^*_{k+1} \in \mathfrak{D}^0$, since $\mathfrak{D} \subset \mathfrak{D}^0$. Hence, by (7.13),

$$u^*_{k+1} = H^k u_1 = (H^0)^k u_1 = \sum_{a=1}^{\infty} \sqrt{\mu_a}\, \lambda_a{}^k \phi_a. \tag{7.21}$$

Thus we obtain as the condition for the existence of the element u^*_{k+1}

$$\sum_{a=1}^{\infty} \mu_a \lambda_a{}^{2k} < \infty \quad \text{for } k = 1, 2, \cdots. \tag{7.22}$$

On the other hand, u^*_{k+1} belongs to $\mathfrak{D}$, if and only if

$$(H^0{}_x u^*_{k+1}, \Phi(\bar{x})) = 0.$$

Hence, by (7.21), (2.12), (2.17) and (1.21),

$$\sum_{a=1}^{\infty} \mu_a q'(\lambda_a) \Omega_a{}^2 \lambda_a{}^k = 0, \qquad (k = 0, 1, 2, \cdots)$$

or, by (2.16),

$$\sum_{a=1}^{\infty} \frac{\lambda_a{}^k}{q'(\lambda_a)} = 0, \qquad (k = 0, 1, 2, \cdots). \tag{7.23}$$

The equations (7.12), (7.22), (7.23) give, therefore, the necessary and sufficient conditions that the sequence of elements

$$u^*_1 = u_1,\ u^*_2 = Hu_1,\ u^*_3 = H^2 u_1, \cdots, u^*_{k+1} = Hu^*_k = H^k u_1, \cdots$$

be contained in $\mathfrak{D}$.

7.3. We now investigate whether this sequence of elements satisfies the equations (5.212). We have

$$(\phi^*(x), u^*_{k+1}) = (\phi^*(x), H^k u_1) = (H^{*k}\phi^*(x), u_1) = x^k(\phi^*(x), u_1).$$

Hence

$$(\phi^*(x), u^*_{k+1}) = x^k \quad \text{for } k = 0, 1, 2, \cdots \tag{7.31}$$

if, and only if,

$$(\phi^*(x), u_1) = 1.$$

On the other hand, we obtain from (7.13), (2.12) and (2.14)

$$(\phi^*(x), u_1) = \sum_{\alpha=1}^{\infty} \mu_\alpha(\phi^*(x), \phi^*_\alpha) = \sum_{\alpha=1}^{\infty} \mu_\alpha \frac{q(x)v(\lambda_\alpha)}{x - \lambda_\alpha}$$

and this equals 1, if and only if, by (2.16),

$$\frac{1}{q(x)} = \sum_{\alpha=1}^{\infty} \mu_\alpha \frac{v(\lambda_\alpha)}{x - \lambda_\alpha} = \sum_{\alpha=1}^{\infty} \frac{1}{q'(\lambda_\alpha)(x - \lambda_\alpha)}. \tag{7.32}$$

Thus (7.32) gives the necessary and sufficient condition that (7.31) holds. Moreover, we notice readily that (7.32) implies (6.12).

7.4. By E. Schmidt's familiar procedure we determine a sequence of real coefficients o_{nk} $(n = 1, 2, \cdots \to \infty,\ 1 \leq k \leq n)$, such that the elements

$$u_n = \sum_{k=1}^{n} o_{nk} u^*_k$$

form an orthogonal system. Then we obtain from (7.31)

$$(\phi^*(x), u_n) = \sum_{k=1}^{n} o_{nk} x^{k-1} = G_n(x), \quad G_1(x) = 1. \tag{7.41}$$

We have, moreover, by (2.12),

$$\sum_{\alpha=1}^{\infty} \mu_\alpha(t) G_n(\lambda_\alpha(t)) G_m(\lambda_\alpha(t)) = \sum_{\alpha=1}^{\infty} (u_n, \phi_\alpha(t))(\phi_\alpha(t), u_m) = (u_n, u_m) = \delta_{nm} \tag{7.42}$$

Since, by (7.41), $G_n(x)$ is a polynomial of degree $n - 1$ the equations (7.42) mean that the sequence of polynomials is an orthonormal set with respect to the mass distribution $\mu_\alpha(t)$ at the points $\lambda_\alpha(t)$.

We readily show that the orthonormal set $\{u_n\}$ is complete; for the assumption that there is an element w of $\mathfrak{H}$ with $(w, u_n) = 0$ for every n implies, by (2.12) and (7.41), that

$$0 = \sum_{\alpha=1}^{\infty} (w, \phi_\alpha)(\phi_\alpha, u_n) = \sum_{\alpha=1}^{\infty} (w, \phi^*_\alpha)\mu_\alpha G_n(\lambda_\alpha),$$

and from this follows, by a theorem of M. Riesz,[40]

$$\sum_{\alpha=1}^{\infty} \mu_\alpha |(w, \phi^*_\alpha)|^2 = \sum_{n=1}^{\infty} \left| \sum_{\alpha=1}^{\infty} \mu_\alpha (w, \phi^*_\alpha) G_n(\lambda_\alpha) \right|^2 = 0,$$
$$(w, \phi_\alpha) = 0 \quad (\alpha = 1, 2, \cdots), \quad w = 0.$$

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[40] [7], p. 223. See also [8], p. 583, Theorem 10.40.

The equations (4.21), (4.22) and (7.41) lead to

$$(7.43) \quad (R_x{}^t u_n, u_m) = \sum_{a=1}^{\infty} \frac{\mu_a(t)}{\lambda_a(t) - x} G_m(\lambda_a(t)) G_n(\lambda_a(t)),$$

$$(7.44) \quad (H^t u_n, u_m) = a_{m,n} = \sum_{a=1}^{\infty} \mu_a(t)\lambda_a(t) G_m(\lambda_a(t)) G_n(\lambda_a(t)).$$

A familiar argument implies that we can put

$$xG_m(x) = b_1 G_1(x) + \cdots + b_{m+1} G_{m+1}(x),$$

which we substitute in (7.44). Then (7.44) yields, by (7.42),

$$a_{m,n} = 0 \text{ for } m \leq n - 2, \quad a_{m,n} = a_{n,m}.$$

Hence $J = (a_{m,n})$ is a Jacobi matrix with real elements.

In a similar way we put

$$x^k = \sum_{n=1}^{k+1} s_{k+1,n} G_n(x), \quad s_{k+1,1} = c_k, \quad s_{11} = c_0 = 1, \qquad (k = 0, 1, \cdots).$$

Then we obtain from (7.42)

$$(7.45) \quad \sum_{a=1}^{\infty} \mu_a(t)\lambda_a{}^k(t) = \sum_{n=1}^{k+1} s_{k+1,n} \sum_{a=1}^{\infty} \mu_a(t) G_1(\lambda_a(t)) G_n(\lambda_a(t)) = s_{k+1,1} = c_k, \qquad (k = 0, 1, \cdots),$$

which shows that the sequence $\{c_k\}$ defines an undetermined moment problem. The solutions of this problem given in (7.45) are all maximal distributions of masses, since, by (7.43), every solution (7.45) is associated with a resolvent $R_x{}^t$.

By putting in (7.43) $u_n = u_m = u_1$, we are led by (7.45) to

$$(R_x{}^t u_1, u_1) = \sum_{a=1}^{\infty} \frac{\mu_a(t)}{\lambda_a(t) - x} = \frac{p(x) + tu(x)}{q(x) + tv(x)} \sim -\sum_{n=0}^{\infty} \frac{c_k}{x^{k+1}}, \quad c_0 = 1,$$

and this asymptotic power series coincides with (5.43).

7.5. We summarize the result of this section in

THEOREM 3. *Let H be a c.H.p.t. of d.i. (1,1) which satisfies the hypothesis of Theorem 1, and let $p(x)$, $q(x)$, $u(x)$, $v(x)$ be the integral functions defined in Theorem 1. In order that a complete orthonormal set*

of elements u_n can be determined, such that $(Hu_n, u_m) = a_{m,n}$ is the element of a Jacobi matrix J, it is necessary and sufficient that the conditions (7.12), (7.22), (7.23) *and* (7.32) *are satisfied for*

$$\mu_a = \frac{1}{q'(\lambda_a)v(\lambda_a)}.$$

8. The construction of all undetermined moment problems.

8.1. We first show that the condition (7.23) in Theorem 2 can be replaced by the less strict condition

$$\sum_{a=1}^{\infty} \frac{\lambda_a{}^k}{q'(\lambda_a)} < \infty, \qquad (k = 0, 1, 2, \cdots \to \infty), \tag{8.11}$$

by proving

LEMMA 2. *Let $g(x)$ be a transcendental integral function of finite order. If $1/g(x)$ can be represented for every $\delta > 0$ in the sectors*

$$\delta \leq \arg x \leq \pi - \delta \quad \textit{and} \quad \pi + \delta \leq \arg x \leq 2\pi - \delta \tag{8.12}$$

by the asymptotic series

$$\frac{1}{g(x)} \sim \sum_{n=0}^{\infty} \frac{a_n}{x^n}, \tag{8.13}$$

then

$$a_n = 0 \qquad (n = 0, 1, 2, \cdots).$$

Proof. We assume that a_m is the first coefficient $\neq 0$, so that the supposition (8.13) implies

$$\lim_{x\to\infty} \frac{x^m}{g(x)} = a_m \neq 0 \tag{8.14}$$

within the two sectors (8.12). Let $\omega_1, \omega_2, \cdots, \omega_m$ be m roots of $g(x)$, such that

$$g^*(x) = \frac{g(x)}{\prod_{\mu=1}^{m} (x - \omega_\mu)}$$

is an integral function of finite order. Then we obtain from (8.14)

$$\lim_{x\to\infty} g^*(x) = 1/a_m \tag{8.15}$$

within the two sectors (8.12). This implies, however, that $g^*(x)$ is bounded within the two sectors (8.12) and also along the straight lines

$$x = re^{i\delta}, \quad x = re^{-i\delta}, \quad x = re^{i(\pi-\delta)}, \quad x = re^{i(\pi+\delta)}.$$

If ρ is the order of $g^*(x)$, we take $2\delta < \pi/\rho$. Then, by a familiar theorem of Phragmén-Lindelöf,[41] $g^*(x)$ is also bounded within the two sectors

$$-\delta \leq \arg x \leq \delta, \qquad \pi - \delta \leq \arg x \leq \pi + \delta.$$

Hence $g^*(x)$ is bounded in the whole complex plane, which implies that $g^*(x)$ is a constant, and $g(x)$ a polynomial of the m-th degree, in contradiction to the supposition that $g(x)$ is a transcendental function. This completes the proof.

Since (8.11) implies that

$$\frac{1}{q'(\lambda_a)} = O\left(\frac{1}{\lambda_a{}^k}\right) \text{ for every } k,$$

the series (8.11) is absolutely convergent, because of $\sum_{a=1}^{\infty} (1/\lambda_a{}^2) < \infty$, $q(x)$ being of order 1, as we stated in **5.3**. Hence, by (7.32),

$$\frac{1}{q(x)} \sim \sum_{k=0}^{\infty} \frac{q_k}{x^{k+1}}, \qquad q_k = \sum_{a=1}^{\infty} \frac{\lambda_a{}^k}{q'(\lambda_a)}$$

within the two sectors (8.12), and, moreover, by Lemma 2, $q_k = 0$. Hence (8.11) implies (7.23).

8.2. We now give the necessary and sufficient conditions for the construction of all undetermined moment problems.

THEOREM 4. *We consider the class $\mathfrak{A}$ of all integral functions of finite order, real for real x, whose roots λ_a are all real and simple, and which satisfy the two conditions*

$$\frac{1}{q(x)} = \sum_{a=1}^{\infty} \frac{1}{q'(\lambda_a)(x - \lambda_a)}, \tag{8.21}$$

$$\sum_{a=1}^{\infty} \frac{\lambda_a{}^k}{q'(\lambda_a)} < \infty \qquad (k = 0, 1, 2, \cdots \to \infty). \tag{8.22}$$

Then we find all sequences $\{c_\nu\}$ defining an undetermined moment problem by associating with any $q(x)$ of class $\mathfrak{A}$ a sequence $\{\mu_a\}$, $(\mu_a > 0)$, such that

$$\sum_{a=1}^{\infty} \mu_a = 1, \quad \sum_{a=1}^{\infty} \mu_a \lambda_a{}^{2k} < \infty \qquad (k = 1, 2, \cdots) \tag{8.23}$$

[41] See e.g. [10], p. 177, § 5.61.

$$(8.24)\quad \sum_{a=1}^{\infty} \frac{1}{\mu_a(q'(\lambda_a))^2} = \infty, \quad \sum_{a=1}^{\infty} \frac{1}{\mu_a\lambda_a^2(q'(\lambda_a))^2} < \infty.$$

If we put

$$(8.25)\qquad c_k = \sum_{a=1}^{\infty} \mu_a\lambda_a^k \qquad (k = 0, 1, 2, \cdots)$$

the sequence $\{c_k\}$ *defines an undetermined moment problem, and the solution given in* (8.25) *is a maximal distribution of masses.*

Proof. Since (8.21), (8.22) and (8.23) coincide with the conditions (7.32), (7.23), (7.12) and (8.11), (8.11) replacing (7.22)), we deduce from Theorem 3 that these are necessary conditions. We see, moreover, that the conditions (8.24) are also necessary by substituting in (1.22) for Ω_a its value given by (2.16).

We now suppose that the conditions (8.21), (8.22), (8.23) and (8.24) are satisfied, and we shall show that the sequence of constants (8.25) defines an undetermined moment problem.

Considering that $q(x)$ is uniquely determined by its roots λ_a and by (8.21), save for a constant factor, we choose this factor such that

$$(8.26)\qquad \sum_{a=1}^{\infty} \frac{1}{(\lambda_a^2+1)\mu_a(q'(\lambda_a))^2} = 1,$$

where the series converges by (8.24). We now determine a c. H. p. t. H of d. i. (1, 1) in the way described in **1** by taking as characteristic values of the self-adjoint extension H^0 the roots of $q(x)$ and by putting in (1.21)

$$\Omega_a = \frac{1}{\sqrt{\mu_a}\,|\,q'(\lambda_a)\,|}.$$

Then the conditions (1.22) are satisfied by (8.24) and (8.26). Since, moreover, the conditions of Theorem 3 are fulfilled by this H, because of (8.21), (8.22), (8.23) and Lemma 2, H can be carried into a Jacobi matrix of d. i. (1, 1), which defines an undetermined moment problem by (7.45). The fact that (8.25) coincides with (7.45) for $t = 0$ leads to the desired result.

8.3. *Remarks on Theorem* 4. We shall give in the Appendix some sufficient conditions for $q(x)$ to belong to the class $\mathfrak{A}$. On the other hand, if $q(x)$ is any function of the class $\mathfrak{A}$, then we readily verify that (8.23) and (8.24) hold for any sequence of positive numbers $\{\mu_a\}$ for which

$$\frac{\kappa_1}{(q'(\lambda_a))^2} \leq \mu_a \leq \frac{\kappa_2}{(q'(\lambda_a))^2},$$

κ_1 and κ_2 being positive constants.

APPENDIX.

Remarks on the integral functions of the class $\mathfrak{A}$.

9.1. We write $q(x) \in \mathfrak{A}$, if $q(x)$ is an integral function whose roots are all real and simple, which is real for real x, and which satisfies the conditions (8.21) and (8.22).

THEOREM 5a. *Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_a < \cdots \to \infty$ be an infinite sequence of positive numbers, $n(r)$ the number of λ_a for which $\lambda_a \leq r$. If*

$$(9.11) \qquad n(r) \sim \gamma r^\rho, \qquad (0 < \rho < \tfrac{1}{2}, \gamma > 0),$$

$$q(x) = \prod_{a=1}^{\infty} (1 + x/\lambda_a),$$

then $q(x) \in \mathfrak{A}$.

Proof. By a result of Professor Titchmarsh,[42] the hypothesis (9.11) implies, even if $0 < \rho < 1$,

$$\lim_{r\to\infty} \frac{\log q(re^{i\theta})}{r^\rho} = \frac{\pi\gamma\, e^{i\theta\rho}}{\sin \pi\rho}, \qquad (-\pi < \theta < +\pi),$$

$$\frac{\log | q(-r_n)|}{r_n{}^\rho} > \pi\gamma \cot \pi\rho - \epsilon$$

for an infinite sequence of numbers $0 < r_1 < r_2 < \cdots < r_n < \to \infty$. This yields

$$(9.12) \qquad \frac{r^k}{| q(re^{i\theta})|} \leq \kappa \exp\{-(\pi\gamma \frac{\cos \theta\rho}{\sin \pi\rho} - \epsilon) r^\rho\}$$

within the sector $-\pi + \delta \leq \theta \leq \pi - \delta$, κ being a suitably chosen positive constant, depending on k, and

$$(9.13) \qquad \frac{r_n{}^k}{(q(-r_n)|} \leq \exp\{-(\pi\gamma \cot \pi\rho - \epsilon) r_n{}^\rho.$$

[42] [9], p. 191, Theorem III; [10], pp. 271-272, § 8.64.

If $0 < \rho < \frac{1}{2}$, we obtain from (9.12) and (9.13), by a familiar argument based on Cauchy's calculus of residues, that

$$\frac{x^k}{q(x)} = \sum_{\alpha=1}^{\infty} \frac{(-\lambda_\alpha)^k}{q'(-\lambda_\alpha)(x+\lambda_\alpha)}.$$

This leads to (8.21) for $k = 0$ and to (8.22), which is the desired result.

9.2. THEOREM 5b. *Let $\{\lambda_\alpha\}$ and $\{\lambda'_\alpha\}$ be two infinite sequences of positive numbers, $n(r) \sim \gamma r^\rho$, $n'(r) \sim \gamma' r^{\rho'}$, where $n(r)$ and $n'(r)$ denote the number-functions corresponding to $\{\lambda_\alpha\}$ and $\{\lambda'_\alpha\}$, respectively. We put*

$$(9.21)\quad q_1(x) = \prod_{\alpha=1}^{\infty}(1 + x/\lambda_\alpha),\ q_2(x) = \prod_{\alpha=1}^{\infty}(1 - x/\lambda'_\alpha),\ q(x) = q_1(x)q_2(x).$$

Then $q(x) \epsilon \mathfrak{A}$

(i) *for $\rho \neq \rho'$, if $0 < \rho < \frac{1}{2}$, $0 < \rho' < \frac{1}{2}$,*

(ii) *for $\rho = \rho'$, $\gamma \neq \gamma'$, if $0 < \tan^2(\pi/2)\rho < \dfrac{\gamma+\gamma'}{|\gamma - \gamma'|}$,*

(iii) *for $\rho = \rho'$, $\gamma = \gamma'$, if $0 < \rho < 1$.*

Proof. Instead of (9.12) and (9.13), $q_2(x)$, for $0 < \rho' < 1$, satisfies the inequalities

$$\frac{r^k}{|q_2(re^{i\theta})|} \leq \kappa' \exp\{-(\pi\gamma' \frac{\cos(\pi - |\theta|)\rho'}{\sin \pi\rho} - \epsilon)r^{\rho'}\}, \quad (\delta \leq |\theta| \leq \pi),$$

$$\frac{r'^k_n}{|q_2(r'_n)|} \leq \exp\{-(\pi\gamma' \cot \pi\rho' - \epsilon)r'^{\rho'}_n\}$$

for a suitably chosen infinite sequence of constants $0 < r'_1 < r'_2 < \cdots \to \infty$. We obtain from this and from (9.12), (9.13) and (9.21)

$$(9.22)\quad \frac{r^k}{|q(re^{i\theta})|} \leq \kappa'' \exp\{-(\pi\gamma \frac{\cos\theta\rho}{\sin\pi\rho} - \epsilon)r^\rho - (\pi\gamma' \frac{\cos(\pi - |\theta|)\rho'}{\sin\pi\rho'} - \epsilon)$$

for $\delta \leq |\theta| \leq \pi - \delta$,

$$(9.23)\quad \frac{r'^k_n}{|q(r'_n)|} \leq \kappa \exp\{-\left(\frac{\pi\gamma}{\sin\pi\rho} - \epsilon\right)r'^\rho_n - (\pi\gamma' \cot\pi\rho' - \epsilon)r'^{\rho'}_n\},$$

$$(9.24)\quad \frac{r^k_n}{|q(-r_n)|} \leq \kappa' \exp\{-(\pi\gamma \cot\pi\rho - \epsilon)r^\rho_n - \left(\frac{\pi\gamma'}{\sin\pi\rho'} - \epsilon\right) r^{\rho'}_n\}.$$

In the case $\rho \neq \rho'$, $0 < \rho < \frac{1}{2}$, $0 < \rho' < \frac{1}{2}$, we derive from these inequalities

$$\frac{x^k}{q(x)}=\sum_{a=1}^{\infty}\frac{(-\lambda_a)^k}{q'(-\lambda_a)(x+\lambda_a)}+\sum_{a=1}^{\infty}\frac{\lambda'^k_a}{q'(\lambda'_a)(x-\lambda_a)}, \tag{9.25}$$

which proves the assertion (i).

In the case $\rho=\rho'$, $\gamma\neq\gamma'$ we have

$$\begin{aligned}\gamma\cos\theta\rho+\gamma'\cos(\pi-|\theta|)\rho&=(\gamma+\gamma')\cos(\pi/2)\rho\cos(\pi/2-|\theta|)\rho\\ &\quad+(\gamma-\gamma')\sin(\pi/2)\rho\sin(\pi/2-|\theta|)\rho\end{aligned} \tag{9.26}$$

Hence $\gamma\cos\theta\rho+\gamma'\cos(\pi-|\theta|)\rho>0$ for $0\leq|\theta|\leq\pi$, if

$$\frac{\gamma+\gamma'}{|\gamma-\gamma'|}>\tan^2(\pi/2)\rho. \tag{9.27}$$

This leads by (9.22), (9.23) and (9.24) again to (9.25), if (9.27) is satisfied, and, as we have shown above, (9.25) implies that $q(x)\in\mathfrak{A}$.

In the case $\rho=\rho'$, $\gamma=\gamma'$ (9.26) can be replaced by

$$\cos\theta\rho+\cos(\pi-|\theta|)\rho=2\cos(\pi/2)\rho\cos(\pi/2-|\theta|)\rho,$$

which implies that, in this case, (9.25) holds for $0<\rho<1$. This leads to the assertion (iii) and completes the proof.

9.3. We now consider the difference

$$\frac{u(x)}{v(x)}-\frac{p(x)}{q(x)}=\frac{1}{v(x)q(x)}$$

where $p(x)$, $q(x)$, $u(x)$, $v(x)$ are the functions (5.35), which are associated with an undetermined moment problem.

On the other hand, we obtain from (5.41) and (5.42) an expansion of $u(x)/v(x)$ and $p(x)/q(x)$ in a series of partial fractions

$$\frac{1}{v(x)q(x)}=\sum_{a=1}^{\infty}\left(\frac{\mu'_a}{x-\lambda'_a}-\frac{\mu_a}{x-\lambda_a}\right); \tag{9.31}$$

here we use the notation

$$\lambda'_a=\lim_{t\to\infty}\lambda_a(t),\qquad \mu'_a=\lim_{t\to\infty}\mu_a(t).$$

Since, by (7.45),

$$c_k=\sum_{a=1}^{\infty}\mu_a\lambda_a^k=\sum_{a=1}^{\infty}\mu'_a\lambda'^k_a,$$

we have, by (9.31), for $k=1,2,\cdots\to\infty$

$$\frac{x^k}{v(x)q(x)} - \sum_{a=1}^{\infty}\left(\frac{\mu'_a\lambda'^k_a}{x-\lambda'_a} - \frac{\mu_a\lambda_a^k}{x-\lambda_a}\right) = \sum_{a=1}^{\infty}\left(\mu'_a\,\frac{x^k-\lambda'^k_a}{x-\lambda'_a} - \mu_a\,\frac{x^k-\lambda_a^k}{x-\lambda_a}\right)$$

$$(9.32) \qquad = \sum_{\kappa=0}^{k-1} x^{k-1-\kappa}\sum_{a=1}^{\infty}(\mu'_a\lambda'^{\kappa}_a - \mu_a\lambda_a^{\kappa}) = 0.$$

The equations (9.31) and (9.32), however, imply that $v(x)q(x) \in \mathfrak{A}$.

9.4. We now denote by λ_a' and λ_a'', respectively, the roots of $q(x)$ for which $q'(\lambda_a') > 0$, $q'(\lambda_a'') < 0$, and put $(q(x) = q_1(x)q_2(x)$, where $q_1(x)$ and $q_2(x)$ are integral functions having the roots λ_a' and λ_a'', respectively. Moreover we write

$$M_a' = \frac{1}{q'(\lambda_a')} = \frac{1}{q_2(\lambda_a')q_1'(\lambda_a')}, \quad M_a'' = -\frac{1}{q'(\lambda_a'')} = -\frac{1}{q_1(\lambda_a'')q_2'(\lambda_a'')}.$$

Then we see, since

$$\sum_{a=1}^{\infty}\frac{\lambda_a^k}{q'(\lambda_a)} = \sum_{a=1}^{\infty} M_a'\lambda_a'^k - \sum_{a=1}^{\infty} M_a''\lambda_a''^k = 0,$$

that the sequence

$$(9.41) \qquad c_k' = \sum_{a=1}^{\infty} M_a'\lambda_a'^k = \sum_{a=1}^{\infty} M_a''\lambda_a''^k$$

defines a new undetermined moment problem. Its solutions given in (9.41), however, are not necessarily maximal distributions, as an example will show in the next paragraph.

9.5. We take two different infinite sequences of constants $\{c_n\}$ and $\{c^*_n\}$ which both may define an undetermined moment problem. Let $p(x)$, $q(x)$, $u(x)$, $v(x)$ be the functions defined in (5.35) which belong to the continued fraction determined by the power series $\sum_{\nu=0}^{\infty}(c_\nu/x^{\nu+1})$ and let $p^*(x)$, $q^*(x)$, $u^*(x)$, $v^*(x)$ be the functions belonging to $\sum_{\nu=0}^{\infty}(c^*_\nu/x^{\nu+1})$. We now consider the two meromorphic functions

$$(9.51) \quad \frac{p_1(x)}{q_1(x)} = \frac{p(x)q^*(x)+u(x)p^*(x)}{q(x)q^*(x)+v(x)p^*(x)}, \quad \frac{p_2(x)}{q_2(x)} = \frac{p(x)v^*(x)+u(x)u^*(x)}{q(x)v^*(x)+v(x)u^*(x)}$$

An easy expansion leads, by (5.39), to

$$(9.52) \quad p_2(x)q_1(x) - p_1(x)q_2(x) = (u(x)q(x) - v(x)p(x))(u^*(x)q^*(x) - p^*(x)v^*(x)) = 1;$$

hence

$$\frac{p_2(x)}{q_2(x)} - \frac{p_1(x)}{q_1(x)} = \frac{1}{q_1(x)q_2(x)}. \tag{9.53}$$

If λ_a' denote the roots of $q_1(x)$, λ_a'' the roots of $q_2(x)$, we have by a theorem of R. Nevanlinna [43]

$$\frac{p_1(x)}{q_1(x)} \sim -\sum_{\nu=0}^{\infty} \frac{c_\nu}{x^{\nu+1}}, \quad \frac{p_2(x)}{q_2(x)} \sim -\sum_{\nu=0}^{\infty} \frac{c_\nu}{x^{\nu+1}},$$

$$\frac{p_1(x)}{q_1(x)} = \sum_{a=1}^{\infty} \frac{M_a'}{\lambda_a' - x}, \quad \frac{p_2(x)}{q_2(x)} = \sum_{a=1}^{\infty} \frac{M_a''}{\lambda_a'' - x},$$

$$c_k = \sum_{a=1}^{\infty} M_a'\lambda_a'^k = \sum_{a=1}^{\infty} M_a''\lambda_a''^k. \tag{9.54}$$

Here the residues M_a' and M_a'' are given by the formulae

$$M_a' = -\frac{p_1(\lambda_a')}{q_1'(\lambda_a')} = \frac{1}{q_2(\lambda_a')(q_1'(\lambda_a')}, \quad M_a'' = -\frac{p_2(\lambda_a'')}{q_2'(\lambda_a'')} = \frac{-1}{q_1(\lambda_a'')q_2'(\lambda_a'')},$$

because of (9.52). It follows, moreover, from (9.53) and (9.54) that $q_1(x)q_2(x) \in \mathfrak{A}$, by the same argument as in **9.3**.

We now see that the equations (9.54) can be derived from the result that $q_1(x)q_2(x) \in \mathfrak{A}$ in the same way as (9.41) in **9.4**. On the other hand, the solutions (9.54) of the moment problem defined by the sequence $\{c_k\}$ are, because of (9.51), not maximal distributions of masses, which are all determined by the meromorphic functions $\dfrac{p(x+tu(x)}{q(x)+tv(x)}$. Thus the equations (9.54) furnish the example mentioned at the end of **9.4**.

9.6. The construction of all infinite sequences $\{c_k\}$ defining an undetermined moment problem can also be derived from the results of **9.3** and **9.4**.

If $q(x)$ is any function of $\mathfrak{A}$, $q(x)$ leads to an undetermined moment problem by the method described in **9.4**. The result of **9.3**, moreover, shows that every undetermined moment problem can be obtained in this way. Unlike the method developed in Theorem 4, however, we cannot determine all the

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[43] [5], p. 33, formula (73). See also [8], p. 577, Theorem 10.38.

solutions of our problem by the representations of the c_k given in (9.41), since these are not necessarily maximal distributions.

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REFERENCES.

[1] H. Hamburger, "Über eine Erweiterung des Stieltjesschen Momentenproblems," I, *Mathematische Annalen*, vol. 81 (1920), pp. 235-319; II, *Mathematische Annalen*, vol. 82 (1921), pp. 120-164; III, *Mathematische Annalen*, vol. 82 (1921), pp. 168-187.

[2] ———, "Contributions to the theory of closed Hermitian transformations of deficiency-index (m, m)" (preliminary note), *Quarterly Journal of Mathematics* (Oxford series), vol. 13 (1942), pp. 117-128.

[3] ———, "Contributions to the theory of closed Hermitian transformations of deficiency-index (m, m)," Part I, *Annals of Mathematics*, vol. 45 (1944), pp. 59-99.

[4] J. von Neumann, "Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren," *Mathematische Annalen*, vol. 102 (1929), pp. 79-131.

[5] R. Nevanlinna, "Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momenten problem," *Annales Accademia scientiarum Fennicae*, Serie A 18 (1922), no. 5.

[6] M. Riesz, "Sur le problème des moments," *Arkiv för Matematik, Astronomi och Fysik*, I, vol. 16 (1922), no. 12; II, vol. 16 (1922), no. 19; III, vol. 17 (1923), no. 16.

[7] ———, "Sur le problème des moments et le théorème de Parseval correspondant," *Acta Litterarum ac Scientiarum, Sectio Scientiarum Mathematicarum, Szeged*, vol. 1 (1922-1923), pp. 209-225.

[8] M. H. Stone, "Linear transformations in Hilbert space and their applications to analysis" (*American Mathematical Society Colloquium Publications*, New York, 1932).

[9] E. C. Titchmarsh, "On integral functions with real negative zeros," *Proceedings of the London Mathematical Society* (2), vol. 26 (1927), pp. 185-200.

[10] ———, *The Theory of Functions* (Oxford, 1932).

A NOTE ON THE LAMBERT TRANSFORM.*

By E. K. Haviland.

Hardy and Littlewood have proved [1] the deep, although Abelian, theorem on power series, that summability in the sense of Lambert implies summability in the sense of Abel. The proof depends on more than the prime number theorem, but implies the prime number theorem. The converse proposition, viz., that Abel summability implies Lambert summability, however, may be shown to be false.[1] If the ordinary power series in $r = \exp(-s)$ are replaced by more general Dirichlet series or corresponding Laplace integrals,[2] there arises an essential difference in that the series, if they converge at all for $r < 1$, need not converge absolutely for $r \leq 1 - \epsilon$. In fact, the Dirichlet series (corresponding to the Stieltjes integrals), even if ordinary Dirichlet series, may possess a strip of conditional and even of non-uniform convergence, while, if they be not ordinary Dirichlet series, they may not have a half-plane of absolute or even of uniform convergence. It therefore becomes of interest to carry out the proof without assuming *absolute*, or even *uniform*, convergence of the integrals involved, and such is the object of the present note. The application of the prime number theorem or, rather, of an extension of the theorem, is exactly the same as in the Hardy-Littlewood case, the modifications being concerned with justifying the somewhat elaborate manipulations which make transcription to the Dirichlet case possible.

Accordingly, let $\alpha(x)$ be a function of bounded variation in $0 \leq x \leq b$, b arbitrarily large, and constant near $x = 0$, say for $0 \leq x \leq a$; and let

$$(1) \qquad A(s) = \int_0^\infty e^{-sx} d\alpha(x),$$

* Received January 22, 1944.

[1] G. H. Hardy and J. E. Littlewood, "On a Tauberian theorem for Lambert's series, and some fundamental theorems in the analytic theory of numbers," *Proceedings of the London Mathematical Society*, ser. 2, vol. 19 (1921), pp. 21-29; cf. also *ibid.*, ser. 2, vol. 41 (1936), pp. 257-270.

[2] A. Wintner, *Eratosthenian Averages* (Baltimore, 1943), pp. 75-76. It should be noted that the power series there referred to should be $\Sigma f'(n) n^{-1} r^n$ and correspondingly the step function should consist of the jumps $\alpha(n+0) - \alpha(n-0) = f'(n)/n$. Also, $sA(s) \to C$ should be replaced by $A(s) \to C$, hence $A(s) = o(1/s)$ by $A(s) = o(1)$.

and

$$(2) \qquad L(s) = \int_0^\infty \frac{xe^{-sx}}{1-e^{-sx}}\,d\alpha(x).$$

We then have the

THEOREM. *The existence of either of the integrals* (1), (2) *implies that of the other, and, if*

$$L(s) = o(s^{-1}),\ \textit{as}\ s \to 0+,\ \textit{then}\ A(s) = o(1),\ \textit{as}\ s \to 0+,$$

there being no actual limitation in supposing $C = 0$, if $\lim sL(s) = C$ as $s \to 0+$.

First of all, we observe that Abel's lemma, which forms the basis of the proof of the second mean value theorem, may be stated in the following form:

Let $f(x)$ be continuous, decreasing and non-negative in $0 < a \leq x < +\infty$; let $\phi(x)$ be bounded and $\alpha(x)$ be of bounded variation in every finite interval, $0 \leq x \leq R$, R arbitrarily large; finally let $\int_a^\infty \phi(x)\,d\alpha(x)$ converge. Then $\int_a^\infty f(x)\phi(x)\,d\alpha(x)$ converges, and

$$(3) \qquad f(a)m \leq \int_a^\infty f(x)\phi(x)\,d\alpha(x) \leq f(a)M,$$

where m and M are respectively the g.l.b. and l.u.b. of $\int_a^x \phi(t)\,d\alpha(t)$ in $a \leq x < +\infty$.

Now let $s > 0$ be fixed. Then

$$d(xe^{-skx})/dx = e^{-skx}(1-skx) < 0, \text{ if } x > 2/s \text{ and } k \geq \tfrac{1}{2}.$$

Consequently, by choosing $\phi(x) = e^{-\frac{1}{2}sx}$, $f(x) = xe^{-(n-\frac{1}{2})sx}$, $(n = 1, 2, 3, \cdots)$, we may infer the existence of

$$(4) \qquad \int_0^\infty xe^{-nsx}\,d\alpha(x).$$

Again, if we choose $\phi(x) = e^{-\frac{1}{2}sx}$ and $f(x) = xe^{-\frac{1}{2}sx}(1-e^{-sx})^{-1}$, it follows from

$$df(x)/dx = [e^{\frac{1}{2}sx}(1-\tfrac{1}{2}sx) - e^{-\frac{1}{2}sx}(1+\tfrac{1}{2}sx)]e^{-sx}(1-e^{-sx})^{-2},$$

that $f(x)$ is monotone decreasing at least for all $x > 2/s$. As before, we then infer the existence and convergence of $L(s)$. Moreover, (1) may be written in the form

$$A(s)=\int_a^\infty \frac{1-e^{-sx}}{x}\cdot\frac{x}{1-e^{-sx}}\,e^{-sx}d\alpha(x),$$

and if we put $f(x)=(1-e^{-sx})x^{-1}$ and $\phi(x)=xe^{-sx}(1-e^{-sx})^{-1}$, it is seen that

$$d\{(1-e^{-sx})x^{-1}\}/dx=(sxe^{-sx}-1+e^{-sx})x^{-2}<0,$$

if $s>0$ and x is sufficiently large. Consequently, the Abel lemma may be applied in this case also and shows that the convergence of $A(s)$ follows from that of $L(s)$.

Having established the existence of the individual integrals involved, if, say, the existence of $A(s)$ is assumed, we proceed to justify the term by term integration

(5) $$L(s)=\int_0^\infty xe^{-sx}\sum_{n=0}^\infty e^{-nsx}d\alpha(x)=\sum_{n=1}^\infty\int_0^\infty xe^{-snx}d\alpha(x).$$

This will be true, if [3]

(i) $\sum_{n=0}^\infty \int_a^x e^{-nst}te^{-st}d\alpha(t)$ converges for x in any finite interval $0<a\leq x\leq R$;

(ii) $\int_a^\infty xe^{-sx}\sum_{k=0}^n e^{-ksx}d\alpha(x)$ converges for n in $0\leq n\leq N$; and

(iii) $\sum_{n=0}^\infty \int_a^x e^{-nst}te^{-st}d\alpha(t)$ converges uniformly in $a\leq x<+\infty$.

The existence of (4) implies that

$$\left|\int_R^{R'}\sum_{k=0}^n e^{-ksx}xe^{-sx}d\alpha(x)\right|<(N+1)\epsilon$$

uniformly for all n, $(0\leq n\leq N)$, if $R_0(\epsilon)\leq R<R'$, which proves (ii).

If $f(x)=e^{-(n-1)sx}$ and $\phi(x)=xe^{-sx}$, (3) leads to the result

(6) $$\left|\int_a^x e^{-(n-1)st}te^{-st}d\alpha(t)\right|\leq e^{-(n-1)sa}M,\qquad (n=2,3,\cdots),$$

where

$$M=\underset{a<x<\infty}{\text{l. u. b.}}\left|\int_a^x te^{-st}d\alpha(t)\right|$$

[3] Cf. W. F. Osgood, *Functions of Real Variables* (Peking, 1936 and New York, 1938), p. 166; H. S. Carslaw, *Fourier Series and Integrals*, 3rd Ed. London (1930), § 76, p. 178.

has been shown to exist, and (6) holds uniformly for $a \leq x < +\infty$. Therefore, $(1+\sum_{n=1}^{\infty} e^{-nsa})M$ serves as a majorant for the series in (i) and in (iii) and the proof of (5) is complete.

Next, application of the Abel lemma to

$$\int_R^{R'} e^{-(s-\sigma)x} e^{-\sigma x} d\alpha(x) \quad \text{and} \quad \int_R^{R'} e^{-(s-\sigma)x} x e^{-\sigma x} d\alpha(x),$$

where $0 < \sigma < s_0 \leq s$, shows that the integrals

$$A(s) = \int_a^{\infty} e^{-sx} d\alpha(x) \quad \text{and} \quad \int_a^{\infty} x e^{-sx} d\alpha(x)$$

converge uniformly in $s_0 \leq s < +\infty$. Then by Leibniz's Rule [4]

(7) $A'(ns) = -\int_a^{\infty} x e^{-nsx} d\alpha(x)$, where $s > 0$, ($n = 1, 2, 3, \cdots$ and $' = d/d(ns)$).

Consequently, (5) leads to the infinite system of equations

$$L(ms) = -\sum_{n=1}^{\infty} A'(mns), \qquad (m = 1, 2, 3, \cdots).$$

These represent a set of equations of the form

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots \\ &\cdots\cdots\cdots\cdots\cdots, \end{aligned}$$

where $y_m = L(ms)$, $x_n = -A'(ns)$ and $(a_{ij}) = (\epsilon_{nm})^*$, the transposed matrix of the Eratosthenian matrix,[5] so det $(a_{ij}) = 1$.

On replacing s in (7) by ms and applying the Abel lemma, we find

$$(8) \quad \begin{aligned} |x_{mn}| = |A'(mns)| &\leq e^{-m(n-1)sa} \operatorname*{l.u.b.}_{a<x<\infty} \int_a^{x} x e^{-msx} d\alpha(x) \\ &\leq e^{-m(n-1)sa} \operatorname*{l.u.b.}_{a<x<\infty} \int_a^{x} x e^{-sx} d\alpha(x) = e^{-m(n-1)sa} M, \text{ say,} \end{aligned}$$

where $n = 2, 3, \cdots$, while

$$(9) \qquad |A'(ms)| \leq e^{-(m-1)sa} M.$$

[4] Cf. W. F. Osgood, *op. cit.*, p. 279.

[5] Cf. A. Wintner, *op. cit.*, pp. 5-7 and p. 23.

By virtue of (8), (9) and the fact that $|a_{ij}| \leq 1$, the series for y_m is majorized by

$$\begin{aligned} Me^{-(m-1)sa} + M \sum_{k=1}^{\infty} e^{-kmsa} &= Me^{-(m-1)sa} + Me^{-msa}(1 - e^{-msa})^{-1} \\ &\leq Me^{-(m-1)sa} + Me^{-msa}(1 - e^{-sa})^{-1}, \end{aligned}$$

where $s > 0$, $a > 0$ are fixed, and $m = 1, 2, 3, \cdots$.

If we multiply y_m by A_{mq}, where A_{ij} is the cofactor of a_{ij} in $\det(a_{ij})$, the resulting series converges and possesses the same majorant, since $A_{mq} = \mu(m/q)$, the Möbius function,[5] and $\mu(m/q) = 0$ unless m/q is an integer. Then in the double series

$$\begin{aligned} \sum_m A_{mq} y_m &= \sum_m \sum_k A_{mq} a_{mk} x_k \\ &= A_{1q} a_{11} x_1 + A_{1q} a_{12} x_2 + \cdots \\ &\quad + A_{2q} a_{21} x_1 + A_{2q} a_{22} x_2 + \cdots \\ &\quad + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad , \end{aligned}$$

not only does each row converge absolutely, but the series of the sums of the absolute values of the elements of the rows likewise converges (absolutely), being majorized by

$$M \sum_{k=0}^{\infty} e^{-ksa} + M(1 - e^{-sa})^{-1} \sum_{k=1}^{\infty} e^{-ksa}.$$

Hence the double series converges and we may sum by columns as well as by rows, obtaining, in particular, $x_1 = \sum_{m=1}^{\infty} A_{m1} y_m$, i. e., on changing the summation letter from m to n,

$$A'(s) = -\sum_{n=1}^{\infty} \mu(n) L(ns).$$

That $A(\infty) = 0$ follows from the definition of $A(s)$. We wish to show that

$$(10) \quad A(s) = \int_s^{\infty} \sum_{n=1}^{\infty} \mu(n) L(nt)\, dt = \sum_{n=1}^{\infty} \mu(n) \int_s^{\infty} L(nt)\, dt, \qquad (s > 0).$$

This will be true if

(I) $\sum_{n=1}^{\infty} \mu(n) \int_s^{x} L(nt)\, dt$ converges for x in any finite interval $s \leq x \leq R$;

(II) $\int_s^{\infty} \sum_{k=1}^{n} \mu(k) L(kt)\, dt$ converges for n in $1 \leq n \leq N_2$, say;

(III) $\sum_{n=1}^{\infty} \mu(n) \int_s^{x} L(nt)\, dt$ converges uniformly for x in $s \leq x < +\infty$.

The proof of these statements is based on methods used above, and which, together with the definition of L, insure the existence of the individual integrals in (10). Here we put

$$(11)\qquad L(s) = \int_a^\infty e^{-\frac{1}{2}sx}\cdot e^{-\frac{1}{2}sx}(1-e^{-sx})^{-1}d\alpha(x).$$

Now

$$\int_a^x te^{-\frac{1}{2}st}(1-e^{-st})^{-1}d\alpha(t) = \int_a^x te^{-\frac{1}{2}(s-\sigma)t}(1-e^{-st})^{-1}\cdot e^{-\frac{1}{2}\sigma t}d\alpha(t),$$

where $0 < \sigma < s_0 \leq s$, and

$$d\{te^{-\frac{1}{2}(s-\sigma)t}(1-e^{-st})^{-1}\}/dt$$
$$= [\{1-\tfrac{1}{2}(s-\sigma)t\}e^{-\frac{1}{2}(s-\sigma)t}(1-e^{-st}) - ste^{-\frac{1}{2}(s-\sigma)t}e^{-st}](1-e^{-st})^{-2},$$

and this expression is certainly negative if $t > \xi = 2/(s_0-\sigma)$. Consequently, if $x > \xi$, an application of the Abel lemma shows that

$$\left|\int_\xi^x te^{-\frac{1}{2}st}(1-e^{-st})^{-1}d\alpha(t)\right| \leq \xi e^{-\frac{1}{2}s_0\xi}(1-e^{-s_0\xi})^{-1}\ \underset{\xi<x<\infty}{\text{l. u. b.}}\left|\int_\xi^x e^{-\frac{1}{2}\sigma t}d\alpha(t)\right| = M_1.$$

Also, if $a \leq x \leq \xi$

$$\left|\int_a^x te^{-\frac{1}{2}st}(1-e^{-st})^{-1}d\alpha(t)\right| \leq \xi e^{-\frac{1}{2}s_0a}(1-e^{-s_0a})^{-1}\int_a^\xi |d\alpha(x)| = M_2,$$

where M_1 and M_2 are independent of s in $s_0 \leq s < +\infty$, and

$$\left|\int_a^x te^{-\frac{1}{2}st}(1-e^{-st})^{-1}d\alpha(t)\right| \leq M_1 + M_2 = M_3,$$

where M_3 is independent of x in $a \leq x < +\infty$.

In view of this, an application of the Abel lemma to (11) shows that

$$(12)\qquad |L(ns)| \leq M_3e^{-\frac{1}{2}nsa}, \qquad (n = 1, 2, 3, \cdots).$$

From this it follows that

$$\left|\int_R^{R'}\mu(n)L(nt)dt\right| \leq M_3\int_R^{R'}e^{-\frac{1}{2}nat}dt = 2M_3(na)^{-1}(e^{-\frac{1}{2}naR}-e^{-\frac{1}{2}naR'})$$
$$< 2M_3a^{-1}e^{-\frac{1}{2}aR},$$

so that (II) is satisfied. Again,

$$\int_s^x L(nt)dt \leq M_3\int_s^x e^{-\frac{1}{2}nat}dt < 2M_3a^{-1}e^{-\frac{1}{2}nas_0},$$

from which it is seen that (I) and (III) are satisfied and (10) is accordingly proved.

From this point on, the proof is merely a transcription to the case of integrals of that of Hardy and Littlewood in the case of power series, but will be given for the sake of completeness. If, in the right-hand member of (10), we make the substitution $nt = \tau$ and then replace τ by t, we obtain

$$A(s) = \sum_{n=1}^{\infty} \mu(n) n^{-1} \int_{ns}^{\infty} L(t)\,dt \equiv \int_0^{\infty} \{ \int_{sx}^{\infty} L(t)\,dt \} d\lambda(x), \tag{13}$$

if $\lambda(x)$ denotes the $[x]$-th partial sum of the series $\Sigma \mu(n)/n$. Let $f(x) = \int_{sx}^{\infty} L(t)\,dt$. Then a partial integration shows that

$$\left| \int_{R'}^{R''} \lambda(x)\,df(x) \right| \leq \left| \int_{R'}^{R''} f(x)\,d\lambda(x) \right| + |f(R'')\lambda(R'')| + |f(R')\lambda(R')|. \tag{14}$$

Of the three terms on the right of (14), the first can be made arbitrarily small by taking R' sufficiently large, in virtue of the convergence of the integral on the right of (13). Furthermore, $\lambda(x)$ remains bounded as $x \rightarrow \infty$, inasmuch as the prime number theorem implies the convergence of $\Sigma \mu(n)/n$ (and that, too, to 0), and $f(R')$, $f(R'') \rightarrow 0$ as $R' \rightarrow +\infty$ in view of the definition of $f(x)$ and the above appraisal of $L(t)$. Consequently, $\int_0^{\infty} \lambda(x)\,df(x)$ converges, and if we let R become infinite in the equation

$$\int_0^R f(x)\,d\lambda(x) = f(R)\lambda(R) - f(0)\lambda(0) - \int_0^R \lambda(x)\,df(x)$$

and observe that $\lambda(0) = 0$, we obtain

$$\begin{aligned} A(s) &= \int_0^{\infty} f(x)\,d\lambda(x) = -\int_0^{\infty} \lambda(x)\,df(x) = s \int_0^{\infty} \lambda(x) L(sx)\,dx \\ &= \int_0^{\infty} \lambda(x/s) L(x)\,dx = \int_0^{\delta} + \int_{\delta}^{1} + \int_1^{\infty} = \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Now

$$|\mathrm{III}| \leq \int_1^{\infty} |\lambda(x/s)|\ |L(x)|\,dx$$

and the prime number theorem asserts the convergence of $\Sigma \mu(n)/n$ to 0, so that $\lambda(x) = o(1)$ as $x \rightarrow \infty$, i.e., $|\lambda(x)| < \epsilon$, if $x \geq x_0 = x_0(\epsilon)$. Then we can choose σ_1 sufficiently small that $s^{-1} > x_0$ for all $0 < s < \sigma_1$. This fact,

together with (12), shows that $| \text{III} | < \epsilon M'_3 \int_1^\infty e^{-x} dx$ for a suitable constant M'_3, so III is $o(1)$ as $s \to 0$.

By virtue of the hypothesis $L(x) = o(x^{-1})$, there exists a $\delta = \delta(\epsilon)$ such that $| L(x) | \leq \epsilon / x$ for $0 < x \leq \delta$. Moreover, for fixed $s > 0$, $\lambda(x/s) = 0$ in $0 \leq x < s$. Consequently,

$$| \text{I} | \leq \epsilon \int_s^\delta | \lambda(x/s) | x^{-1} dx = \epsilon \int_1^{\delta/s} | \lambda(y) | y^{-1} dy.$$

Now $\lambda(y)$ is always bounded and, as $y \to \infty$, it is known that $\lambda(y) = o(\log^{-\alpha} y)$ for every $\alpha \geq 0$ and so for some $\alpha > 1$; hence *a fortiori*

$$| \lambda(y) | \leq K'_2 \log^{-\alpha} y, \quad \text{if} \quad y > y_0 > 1, \text{ say.} \tag{15}$$

Then

$$\begin{aligned} | \text{I} | &\leq \epsilon \{ \int_1^{y_0} | \lambda(y) | y^{-1} dy + K'_2 \int_{y_0}^{\delta/s} \log^{-\alpha} y \, dy/y \} \\ &< \epsilon \{ K_1 + K_2 \log^{1-\alpha} y_0 \} < K\epsilon. \end{aligned}$$

Finally, for fixed δ, $| L(x) | < G_\delta$, a constant depending only on δ, where we may always suppose $0 < \delta < 1$. For this δ, s has been taken so small that $\delta / s > y_0$ and hence by the first mean value theorem, in virtue of (15),

$$| \text{II} | \leq G_\delta \int_\delta^1 | \lambda(x/s) | dx = G_\delta s \int_{\delta/s}^{1/s} | \lambda(y) | dy \leq K_3 \log^{-\alpha}(\delta/s)$$

and this $\to 0$, as $s \to 0+$. Consequently, $\text{I} + \text{II} + \text{III} = A(s) = o(1)$, as $s \to 0+$, q. e. d.

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ON THE THEORY OF AUTOMORPHIC FUNCTIONS OF A MATRIX VARIABLE, II—THE CLASSIFICATION OF HYPERCIRCLES UNDER THE SYMPLECTIC GROUP.*

By Loo-Keng Hua.

1. Introduction. The present paper is a continuation of the paper I with the same title,[1] which gives a brief account of the geometrical aspect of the theory.

Throughout the paper, capital Latin letters denote $n \times n$ matrices with complex elements unless the contrary is stated. A' denotes the transposed matrix of A and $\bar{A}$, the conjugate imaginary matrix of A. I denotes the unit matrix and O, the zero matrix.

We define a hypercircle to be the set of points (symmetric matrices) Z for which the Hermitian matrix

$$\bar{Z}H_1Z + LZ + \bar{Z}\bar{L}' + H_2$$

is positive definite, where H_1 and H_2 are Hermitian matrices.

The object of the present paper is to classify completely hypercircles under the (non-homogeneous) symplectic group $\mathfrak{G}$, which consists of all the symplectic transformations defined by:

$$Z_1 = (AZ + B)(CZ + D)^{-1}, \quad AB' = BA', \quad CD' = DC', \quad AD' - BC' = I.$$

The letter $\mathfrak{G}$ will be kept in this sense throughout the paper.

Our classification of hypercircles depends on the theory of pairs of Hermitian matrices. Because all the available treatments (or at least all the treatments available to the author in China, cf. **6**) of the subject contain a mistake, we find it necessary to resume the theory.

2. Symmetric pairs of matrices. Let

$$\mathfrak{F} = \begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

Note.—Because of the poor mail service between the U. S. and China, a number of minor changes in this paper have been made here, with the consent of the editors, by Prof. Hua's friend Dr. Hsio-Fu Tuan.

* Received April 21, 1943.

[1] This *Journal*, vol. 66 (1944), pp. 470-488.

which is a $2n \times 2n$ skew symmetric matrix. This notation will be kept throughout.

DEFINITION 1. *A pair of matrices A and B is said to be symmetric to each other, or to form a symmetric pair (A,B), if $AB' = BA'$.*

Clearly (A, B) *is a symmetric pair if and only if*

$$(A,B)\mathfrak{F}(A,B)' = O,$$

since the left hand side is equal to

$$(-B,A)(A,B)' = -BA' + AB'.$$

DEFINITION 2. *A pair of matrices (C,D) is said to be conjugate to another pair of matrices (A,B) if $AD' - BC' = I$.*

According to this definition, the conjugate relation is skew in the two pairs: if (C,D) is conjugate to (A,B), then $-(A,B) = (-A,-B)$ is conjugate to (C,D). In the following we shall often speak of conjugate pairs, when the order of the pairs is immaterial.

Clearly (C,D) *is conjugate to* (A,B) *if and only if*

$$(A,B)\mathfrak{F}(C,D)' = I,$$

since the left hand side is exactly $AD' - BC'$.

THEOREM 1. *The transformation*

$$Z_1 = (AZ + B)(CZ + D)^{-1}$$

with the matrix

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(in the following we often speak of the transformation $\mathfrak{T}$) *belongs to* $\mathfrak{G}$, *if and only if*

$$\mathfrak{T}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}.$$

Proof. Since the left hand side of the equation to be proved is

$$\begin{pmatrix} AB' - BA', & AD' - BC' \\ CB' - DA', & CD' - DC' \end{pmatrix},$$

the result follows immediately.

Putting this result in another form, we have:

THEOREM 2. *The transformation*

$$Z_1 = (AZ + B)(CZ + D)^{-1}$$

with the matrix

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

belongs to $\mathfrak{G}$ *if and only if* (A, B) *and* (C, D) *are two symmetric pairs such that* (C, D) *is conjugate to* (A, B).

THEOREM 3. *If* (A, B) *is a symmetric pair, then*

$$(A_1, B_1) = Q(A, B)\mathfrak{T}$$

is also a symmetric pair, where $\mathfrak{T}$ *is in* $\mathfrak{G}$.

Proof. We have

$$\begin{aligned}(A_1, B_1)\mathfrak{F}(A_1, B_1)' &= Q(A, B)\mathfrak{T}\mathfrak{F}\mathfrak{T}'(A, B)'Q' \\ &= Q(A, B)\mathfrak{F}(A, B)'Q' = 0.\end{aligned}$$

THEOREM 4. *If* (A, B) *is conjugate to* (C, D), *and if*

$$(A_1, B_1) = Q(A, B)\mathfrak{T}, \qquad (C_1, D_1) = Q'^{-1}(C, D)\mathfrak{T},$$

then (A_1, B_1) *is conjugate to* (C_1, D_1), *where* Q *is non-singular and* $\mathfrak{T}$ *is in* $\mathfrak{G}$.

Proof. We have

$$(A_1, B_1)\mathfrak{F}(C_1, D_1)' = Q(A, B)\mathfrak{T}\mathfrak{F}\mathfrak{T}'(C, D)'Q^{-1} = I.$$

DEFINITION 3. *Two symmetric pairs of matrices* (A_1, B_1) *and* (A, B) *are said to be equivalent if we have a non-singular matrix* Q *and a transformation* $\mathfrak{T}$ *of* $\mathfrak{G}$ *such that*

$$(A_1, B_1) = Q(A, B)\mathfrak{T}.$$

This relation will be denoted by

$$(A_1, B_1) \sim (A, B).$$

THEOREM 5. *The relation "$\sim$" possesses the properties: determination, reflexivity, symmetry and transitivity.*

DEFINITION 4. *A pair of matrices* (A, B) *is said to be non-singular if the matrix* (A, B) *is of rank* n.

THEOREM 6. *Any two non-singular symmetric pairs of matrices are equivalent.*

Proof. It is sufficient to prove that

$$(A,B) \sim (I,O).$$

1) If A is non-singular, then $A^{-1}B = S$ is symmetric. Then

$$(A,B) = A(I,S) = A(I,O)\begin{pmatrix} I & S \\ O & I \end{pmatrix}.$$

The result follows, since

$$\begin{pmatrix} I & S \\ O & I \end{pmatrix}$$

belongs to $\mathfrak{G}$.

2) If A is singular, then we have two non-singular matrices P and Q such that

$$A_1 = PAQ = \begin{pmatrix} I^{(r)} & O \\ O & O^{(n-r)} \end{pmatrix}.$$

Let

$$(A_1, B_1) = P(A,B)\begin{pmatrix} Q & O \\ O & Q'^{-1} \end{pmatrix},$$

where

$$B_1 = PBQ'^{-1} = \begin{pmatrix} s^{(r)} & m \\ l & t^{(n-r)} \end{pmatrix}, \text{ say.}$$

Since

$$\begin{pmatrix} Q & O \\ O & Q'^{-1} \end{pmatrix}$$

belongs to $\mathfrak{G}$, (A_1, B_1) is a non-singular and symmetric pair. Consequently $s^{(r)}$ is symmetric and l is a null matrix.

Let

$$(A_2, B_2) = (A_1, B_1)\begin{pmatrix} I & -S \\ O & I \end{pmatrix},$$

where

$$S = \begin{pmatrix} s^{(r)} & O \\ O & I^{(n-r)} \end{pmatrix}.$$

Then

$$A_2 = A_1, \qquad B_2 = -A_1S + B_1 = \begin{pmatrix} O & m \\ O & t \end{pmatrix}.$$

Since (A_2, B_2) is non-singular, so also is t. Let

$$(A_3, B_3) = (A_2, B_2)\begin{pmatrix} I & O \\ I & I \end{pmatrix};$$

then

$$A_3 = A_2 + B_2 = \begin{pmatrix} I^{(r)} & m \\ O & t \end{pmatrix}$$

which is non-singular. By 1), we have

$$(A_3, B_3) \sim (I, O).$$

The result follows.

THEOREM 7. *The subgroup which leaves a non-singular symmetric pair of matrices invariant is simply isomorphic to the group which consists of all transformations of the form*

$$Z_1 = Q'ZQ + S,$$

where Q is non-singular and S is symmetric.

Proof. It is sufficient to consider the group which leaves (O, I) invariant. In fact, we have Q and $\mathfrak{T}$ such that

$$Q(A, B)\mathfrak{T} = (O, I).$$

Let Q_0 and $\mathfrak{T}_0$ be such that

$$Q_0(O, I)\mathfrak{T}_0 = (O, I).$$

Then

$$Q^{-1}Q_0Q(A, B)\mathfrak{T}\mathfrak{T}_0\mathfrak{T}^{-1} = (A, B).$$

The isomorphism of the group whose elements leave (O, I) invariant and the group whose elements leave (A, B) invariant is evident.

Let

$$Q(O, I)\mathfrak{T} = (O, I), \qquad \mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then, we have

$$(QC, QD) = (O, I),$$

i. e., $C = O$, $D = Q^{-1}$. Then $A = Q'$ and $B = SQ^{-1}$.

The group is isomorphic to the group formed by the matrices

$$\begin{pmatrix} Q' & SQ^{-1} \\ O & Q^{-1} \end{pmatrix}.$$

The result is now evident.

COROLLARY. *The transformations leaving (O, I) invariant are of the form*

$$\begin{pmatrix} Q' & SQ^{-1} \\ O & Q^{-1} \end{pmatrix},$$

where Q is non-singular and S is symmetric.

THEOREM 8. *Given a non-singular symmetric pair of matrices (A, B), we have a non-singular symmetric pair of matrices (C, D) as its conjugate. The totality of all possible pairs (C, D) depends on $n(n+1)$ parameters.*

Proof. 1) First we consider the case $(A, B) = (O, I)$. Let (C, D) be a pair satisfying our requirement; then

$$I = AD' - BC' = -C'.$$

Thus the conjugate pairs of (A, B) are

$$(-I, S),$$

where the S are symmetric. The theorem is true for $(A, B) = (O, I)$.

2) By Theorem 6, we have Q and $\mathfrak{T}$ such that

$$Q(A, B)\mathfrak{T} = (O, I).$$

We define (C, D) by $Q'^{-1}(C, D)\mathfrak{T} = (-I, S)$. Then (C, D) satisfies our requirement.

Further let Q_1 and $\mathfrak{T}_1$ be matrices satisfying also

$$Q_1(A, B)\mathfrak{T}_1 = (O, I).$$

Then, we have

$$(C, D) = Q'_1(-I, S_1)\mathfrak{T}_1^{-1}.$$

We shall now prove that this is equal to $Q'(-I, S)\mathfrak{T}^{-1}$. Since

$$Q_1Q^{-1}(O, I)\mathfrak{T}^{-1}\mathfrak{T}_1 = (O, I)$$

by the corollary of Theorem 7, we have

$$\mathfrak{T}^{-1}\mathfrak{T}_1 = \begin{pmatrix} Q'^{-1}Q'_1 & S_2QQ_1^{-1} \\ O & QQ_1^{-1} \end{pmatrix}.$$

Then

$$\begin{aligned} Q'_1(-I, S_1)\mathfrak{T}^{-1} &= Q'_1(-I, S_1)\begin{pmatrix} Q'^{-1}Q'_1 & S_2QQ_1^{-1} \\ O & QQ_1^{-1} \end{pmatrix}\mathfrak{T}^{-1} \\ &= Q'_1(-I, S_1)\begin{pmatrix} Q'^{-1}_1Q' & -Q'^{-1}_1Q'S_2 \\ O & Q_1Q^{-1} \end{pmatrix}\mathfrak{T}^{-1} \\ &= Q'(-I, Q'^{-1}Q'_1S_1Q_1Q^{-1} + S_2)\mathfrak{T}^{-1}. \end{aligned}$$

Hence we have always the same collection of pairs of matrices conjugate to (A, B).

3. Hypercircles.

DEFINITION 1. *The transformation of symmetric pairs*

$$(W_1, W_2) = Q(Z_1, Z_2)\mathfrak{T}$$

for a non-singular matrix Q and a transformation $\mathfrak{T}$ belonging to $\mathfrak{G}$ is called a homogeneous representation of

$$W = (AZ + B)(CZ + D)^{-1}.$$

The group so obtained is called the group $\mathfrak{G}_H$.

DEFINITION 2. *A hypercircle is defined by the set of points corresponding to symmetric matrices Z such that the Hermitian matrix*

$$\bar{Z}H_1Z + LZ + \bar{Z}\bar{L}' + H_2$$

is positive definite, where H_1 and H_2 are Hermitian matrices. Or, in "homogeneous" coördinates, a hypercircle is defined by the set of points corresponding to symmetric pairs (W_1, W_2) such that the Hermitian matrix

$$\bar{W}_1H_1W'_1 + \bar{W}_2LW'_1 + \bar{W}_1\bar{L}'W'_2 + \bar{W}_2H_2W'_2 = (\overline{W_1, W_2})\mathfrak{H}(W_1, W_2)'$$

is positive definite, where

$$\mathfrak{H} = \begin{pmatrix} H_1 & \bar{L}' \\ L & H \end{pmatrix}.$$

$\mathfrak{H}$ is called the matrix of the hypercircle.

Remark. $\mathfrak{H}$ is a general $2n \times 2n$ Hermitian matrix. Thus the following results may be interpreted purely algebraically without reference to hypercircles.

THEOREM 9. *The transformation $(W_1, W_2) = Q(Z_1, Z_2)\mathfrak{T}$ carries a hypercircle with the matrix $\mathfrak{H}$ to a hypercircle with the matrix $\mathfrak{H}_1 = \mathfrak{T}\mathfrak{H}\mathfrak{T}'$.*

Proof. Since

$$(\bar{W}_1, \bar{W}_2)\mathfrak{H}(W_1, W_2)' = \bar{Q}(\bar{Z}_1, \bar{Z}_2)\bar{\mathfrak{T}}\mathfrak{H}\mathfrak{T}'(Z_1, Z_2)'Q',$$

the theorem follows.

DEFINITION 3. *If we have $\mathfrak{T}$ belonging to $\mathfrak{G}$ such that $\mathfrak{H}_1 = \bar{\mathfrak{T}}\mathfrak{H}\mathfrak{T}'$, we say that $\mathfrak{H}_1$ and $\mathfrak{H}$ are conjunctive under $\mathfrak{G}$.*

Evidently, "conjunctivity under $\mathfrak{G}$" possesses the properties: symmetry, reflexivity and transitivity. Naturally, this suggests the classification of hyper-

circles under $\mathfrak{G}$. This problem is by no means easy but it is solved completely. First of all, we introduce the following notion:

DEFINITION 4. *For a hypercircle with the matrix $\mathfrak{H}$, we define*

$$\mathfrak{H}'\mathfrak{F}\mathfrak{H} = \begin{pmatrix} H'_1L - L'H_1, & H'_1H_2 - L'\bar{L}' \\ -H'_2H_1 + \bar{L}L, & \bar{L}H_2 - H'_2\bar{L}' \end{pmatrix}$$

to be the discriminantal matrix of the hypercircle. It will be denoted by $\mathfrak{D}(\mathfrak{H})$. Evidently $\mathfrak{D}(\mathfrak{H})$ is skew-symmetric.

THEOREM 10. *If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are conjunctive under $\mathfrak{G}$, then $\mathfrak{D}(\mathfrak{H}_1)$ and $\mathfrak{D}(\mathfrak{H}_2)$ are congruent under $\mathfrak{G}$. More precisely, if $\bar{\mathfrak{T}}\mathfrak{H}_1\mathfrak{T}' = \mathfrak{H}_2$, then*

$$\mathfrak{T}\mathfrak{D}(\mathfrak{H}_1)\mathfrak{T}' = \mathfrak{D}(\mathfrak{H}_2).$$

Proof. Since

$$\mathfrak{D}(\mathfrak{H}_2) = \mathfrak{H}'_2\mathfrak{F}\mathfrak{H}_2 = \mathfrak{T}\mathfrak{H}'_1\bar{\mathfrak{T}}'\mathfrak{F}\bar{\mathfrak{T}}\mathfrak{H}_1\mathfrak{T}' = \mathfrak{T}\mathfrak{H}'_1\mathfrak{F}\mathfrak{H}_1\mathfrak{T}' = \mathfrak{T}\mathfrak{D}(\mathfrak{H}_1)\mathfrak{T}',$$

we have the result.

4. The canonical form of the discriminantal matrix. The problem of congruence of $\mathfrak{D}(\mathfrak{H}_1)$ and $\mathfrak{D}(\mathfrak{H}_2)$ under $\mathfrak{G}$ is equivalent to the problem of congruence of the pairs of skew symmetric matrices $(\mathfrak{D}(\mathfrak{H}_1), \mathfrak{F})$ and $(\mathfrak{D}(\mathfrak{H}_2), \mathfrak{F})$. The latter problem is solved in most treatises on elementary divisors. For the sake of completeness, the author quotes the following results:

THEOREM 11. *Let $\mathfrak{B}$ and $\mathfrak{B}_1$ be two non-singular matrices. The pairs of skew symmetric matrices $(\mathfrak{A}, \mathfrak{B})$ and $(\mathfrak{A}_1, \mathfrak{B}_1)$ are congruent if and only if $\mathfrak{A} + \lambda\mathfrak{B}$ and $\mathfrak{A}_1 + \lambda\mathfrak{B}_1$ have the same invariant factors (or the same elementary divisors).*

(For the proof see, e. g., MacDuffee, *Theory of Matrices*, Theorems 35. 4 and 30. 1.)

THEOREM 12. *There exist pairs of skew symmetric matrices of degree $2n$, one of which is non-singular, having any given admissible invariant factors. More precisely, let*

$$h_{2i} = h_{2i-1} = g_i = (\lambda - \lambda_1)^{l_{i1}} \cdots (\lambda - \lambda_k)^{l_{ik}},$$
$$(1 \leq i \leq n, \quad l_{ij} \geq 0, \quad 1 \leq j \leq k)$$

be the given $2i$-th invariant factors (since in a skew symmetric matrix, the $2i$-th invariant factor is equal to the $(2i\text{-}1)$-th invariant factor), let g_i divide g_{i+1} and let $\Sigma l_{ij} = n$. We define τ_i to be the direct sum of matrices

$$\tau_i = \tau_{i1} \dotplus \tau_{i2} \dotplus \cdots \dotplus \tau_{ik}$$

where

$$\tau_{ij} = \begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ 0 & 0 & \lambda_j & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_j \end{pmatrix}$$

is of degree l_{ij} *and* $1 \leq j \leq h$. *Further we define* T *by the direct sum*

$$T = \tau_n \dotplus \tau_{n-1} \dotplus \cdots \dotplus \tau_{n-t}.$$

Then the pair of skew symmetric matrices $(\mathfrak{E}, \mathfrak{F})$ *with*

$$\mathfrak{E} = \begin{pmatrix} O & T \\ -T' & O \end{pmatrix}, \quad \mathfrak{F} = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}$$

possesses the preassigned invariant factors.

Proof. Let δ_i be the greatest common divisor of the i-rowed minors of $T - \lambda I$. Then, evidently,

$$\delta_i = g_1 \cdots g_i.$$

Further let d_i be the greatest common divisor of the i-rowed minors of $\mathfrak{E} - \lambda\mathfrak{F}$. We need only find the d_i for any even i. It is evident that d_{2i} is the g. c. d. of $\delta_i^2, \delta_{i-1}\delta_{i+1}, \delta_{i-2}\delta_{i+2}, \cdots$. Since

$$\delta_i^2 = g_1^2 \cdots g_i^2, \qquad \delta_{i-t}\delta_{i+t} = g_1^2 \cdots g_{i-t}^2 g_{i-t+1} \cdots g_{i+t},$$

and g_i divides g_{i+1}, we have δ_i^2 dividing $\delta_{i-t}\delta_{i+t}$. Thus $d_{2i} = \delta_i^2$. Then

$$h_{2i}h_{2i-1} = \frac{d_{2i}}{d_{2i-1}} \; \frac{d_{2i-1}}{d_{2i-2}} = \frac{\delta_{i-1}^2}{\delta_i^2}.$$

Since $h_{2i} = h_{2i-1}$, we have

$$h_{2i} = h_{2i-1} = g_i.$$

Consequently we have

THEOREM 13. *Every discriminantal matrix is congruent under* $\mathfrak{G}$ *to a matrix of the form*

$$\begin{pmatrix} O & T \\ -T' & O \end{pmatrix}$$

where T *has the same meaning as given in Theorem 12. Consequently, every*

hypercircle is conjunctive under $\mathfrak{G}$ to a hypercircle with its discriminantal matrix of the prescribed form.

Proof. By Theorems 11 and 12, we have $\mathfrak{T}$ such that

$$\mathfrak{T}\mathfrak{D}(\mathfrak{H})\mathfrak{T}' = \begin{pmatrix} O & T \\ -T' & O \end{pmatrix}$$

and

$$\mathfrak{T}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}.$$

Let $\bar{\mathfrak{T}}^{-1}\mathfrak{H}\mathfrak{T}'^{-1} = \mathfrak{H}_1$; then $\mathfrak{H}_1$ has its discriminantal matrix in the described form.

5. Proof of the theorem that every hypercircle is conjunctive under $\mathfrak{G}$ to a "binomial" hypercircle.

THEOREM 14. *Every hypercircle is conjunctive under $\mathfrak{G}$ to a "binomial" hypercircle, or more precisely a hypercircle with the matrix*

$$\begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}, \quad H_1 = \begin{pmatrix} h_1^{(r)} & O \\ O & O \end{pmatrix}, \quad H_2 = \begin{pmatrix} h_2^{(r)} & O \\ O & O \end{pmatrix}, \quad \det(h_1^{(r)}) \neq 0.$$

Proof. 1) The theorem is well-known for $n = 1$. By Theorem 13, it is sufficient to consider a hypercircle with the matrix

$$\begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix}$$

satisfying the condition

$$\begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix}' \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix} = \begin{pmatrix} O & * \\ * & O \end{pmatrix},$$

i. e., $H'_1L = L'H_1$ and $\bar{L}H_2 = H'_2\bar{L}'$.

If H_1 is non-singular, then

$$S = LH_1^{-1} = \bar{H}_1^{-1}L'$$

is symmetric. We have evidently that

$$\bar{Z}H_1Z + LZ + \bar{Z}\bar{L}' + H_2 = (\bar{Z} + S)H_1(Z + \bar{S}) + H_2 - SH_1\bar{S},$$

which is "binomial" in $Z + \bar{S}$. A similar result holds when H_2 is non-singular. The theorem is thus true for these cases.

2) Before going further, we require two lemmas.

LEMMA 1. *Any symmetric matrix S may be expressed as $S = TT'$ where T is a matrix with zeros above the main diagonal* (well-known).

LEMMA 2. *For any given matrix Q, we have a non-singular symmetric matrix S such that QS is symmetric.*

In fact, it is sufficient to find a non-singular solution of the matrix equation

$$QS = SQ'$$

where the symmetric matrix S is considered as an unknown. We have a non-singular matrix Γ such that

$$Q_1 = \Gamma^{-1}Q\Gamma$$

is of the Jordan's normal form, i. e. a direct sum of matrices of the form

$$J_i^{(j)} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_j \end{pmatrix}.$$

Then

$$Q_1S_1 = S_1Q'_1$$

where $\Gamma^{-1}S\Gamma'^{-1} = S_1$. Therefore, it is sufficient to find a solution of the equation with $Q = J_i^{(j)}$. Evidently

$$S^{(j)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is a solution, since $S = S'$ and

$$J_iS = \begin{pmatrix} 0 & 0 & \cdots & 1 & \lambda_i \\ 0 & 0 & \cdots & \lambda_i & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \lambda_i & \cdots & 0 & 0 \\ \lambda_i & 0 & \cdots & 0 & 0 \end{pmatrix} = SJ'_i.$$

3) We now consider all conjunctive hypercircles of $\mathfrak{H}$ under $\mathfrak{G}$

with "binomial" discriminantal matrices. Let $\mathfrak{H}$ be one of them with H_1 of the highest rank r. If $r=n$, this problem was solved in 1).

We have a non-singular matrix Q such that

$$\bar{Q}'H_1Q=\begin{pmatrix} h^{(r)} & O \\ O & O \end{pmatrix}, \qquad \det(h)\neq 0.$$

Since

$$\mathfrak{T}=\begin{pmatrix} Q' & O \\ O & Q^{-1} \end{pmatrix}$$

carries a hypercircle with "binomial" discriminantal matrix into one of the same nature, we may assume, without loss of generality, that

$$H_1=\begin{pmatrix} h & O \\ O & O \end{pmatrix}, \qquad \det(h)\neq 0, \qquad H_2=\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

We shall now establish that $r\neq 0$ and that we may assume $\det(g_{11})\neq 0$. Let

$$\mathfrak{D}(\mathfrak{H})=\begin{pmatrix} O & K \\ -K' & O \end{pmatrix}$$

be its discriminantal matrix. We may transform $\mathfrak{H}$ in such a way that K is symmetric. In fact, by Lemmas 1 and 2, we have a symmetric matrix S such that (i) SK is symmetric and (ii) $S=TT'$ where T is a matrix with zeros above the main diagonal. Let

$$\mathfrak{T}=\begin{pmatrix} T' & O \\ O & T^{-1} \end{pmatrix}$$

which belongs to $\mathfrak{G}$. Then

$$\mathfrak{T}\mathfrak{D}(\mathfrak{H})\mathfrak{T}'=\begin{pmatrix} O & T'KT'^{-1} \\ -T^{-1}K'T & O \end{pmatrix},$$

where $T^{-1}K'T$ is symmetric, since

$$TT'K=K'TT' \quad \text{implies} \quad T^{-1}K'T=T'KT'^{-1}.$$

Further the first element in $\bar{\mathfrak{T}}\mathfrak{H}\mathfrak{T}'$ is equal to

$$\bar{T}'H_1T=\begin{pmatrix} * & * \\ O & * \end{pmatrix}\begin{pmatrix} h & O \\ O & O \end{pmatrix}\begin{pmatrix} * & O \\ * & * \end{pmatrix}=\begin{pmatrix} * & O \\ O & O \end{pmatrix}$$

of which the rank is still r. We may assume that

$$H_1 = \begin{pmatrix} h & O \\ O & O \end{pmatrix}, \qquad \mathfrak{D}(\mathfrak{H}) = \begin{pmatrix} O & S \\ -S & O \end{pmatrix},$$

where S is symmetric.

Let ρ be any number. We have

$$\begin{pmatrix} I & O \\ \rho I & I \end{pmatrix}\begin{pmatrix} O & S \\ -S & O \end{pmatrix}\begin{pmatrix} I & \rho I \\ O & I \end{pmatrix} = \begin{pmatrix} O & S \\ -S & O \end{pmatrix}$$

and

$$\overline{\begin{pmatrix} I & O \\ \rho I & I \end{pmatrix}}\begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix}\begin{pmatrix} I & \rho I \\ O & I \end{pmatrix} = \begin{pmatrix} H_1 & * \\ * & H_0 \end{pmatrix},$$

where

$$H_0 = |\bar{\rho}|^2 H_1 + \bar{\rho}\bar{L}' = \rho L + H_2.$$

It is evident that for ρ arge, $r = 0$ if and only if $H_1 = L = H_2 = O$. Let

$$H_0 = \begin{pmatrix} k_{11}{}^{(r)} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},$$

then

$$k_{11} = |\rho|^2 h + \rho * + \bar{\rho} * + *.$$

For ρ large, k_{11} is non-singular.

4) Now we may assume that

$$H_1 = \begin{pmatrix} h^{(r)} & O \\ O & O \end{pmatrix}, \qquad H_2 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad L = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}$$

where $\det(h) \neq 0$, $\det(g_{11}) \neq 0$ and $r \neq 0$. Let

$$R = \begin{pmatrix} I & O \\ -g'_{12}\bar{g}_{11}{}^{-1} & I \end{pmatrix}, \qquad \mathfrak{T} = \begin{pmatrix} R'^{-1} & O \\ O & R \end{pmatrix}$$

which belongs to $\mathfrak{G}$, such that

$$\bar{\mathfrak{T}}\mathfrak{H}\mathfrak{T}' = \begin{pmatrix} \bar{R}'^{-1} & O \\ O & \bar{R} \end{pmatrix}\begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix}\begin{pmatrix} R^{-1} & O \\ O & R' \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} h & O \\ O & O \end{pmatrix} & * \\ * & \begin{pmatrix} g & O \\ O & g_0 \end{pmatrix} \end{pmatrix}$$

where $g = g_{11}$. (In fact

$$\bar{R}' H_1 R^{-1} = \begin{pmatrix} I & * \\ O & I \end{pmatrix}\begin{pmatrix} h & O \\ O & O \end{pmatrix}\begin{pmatrix} I & O \\ * & I \end{pmatrix} = \begin{pmatrix} h & O \\ O & O \end{pmatrix},$$

$$\bar{R} H_2 R' = \begin{pmatrix} I & O \\ -\bar{g}'_{12} & g_{11}{}^{1} \end{pmatrix}\begin{pmatrix} g_{11} & g_{12} \\ \bar{g}'_{12} & g_{22} \end{pmatrix}\begin{pmatrix} I & -g_{11}{}^{-1}g_{12} \\ O & I \end{pmatrix} = \begin{pmatrix} g_{11} & O \\ O & g_0 \end{pmatrix}.)$$

Since the rank of H_2 cannot be higher than r, $g_0 = O$.

Now we may write

$$H_1 = \begin{pmatrix} h & O \\ O & O \end{pmatrix}, \qquad L = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}, \qquad H_2 = \begin{pmatrix} g & O \\ O & O \end{pmatrix}$$

where both h and g are non-singular. Since $H'_1 L = \bar{L}' H_1$ and $\bar{L} H_2 = H'_2 \bar{L}'$, we have

$$h' l_{11} = \bar{l}'_{11} h, \qquad \bar{l}_{11} g = g' \bar{l}'_{11}, \qquad l_{12} = l_{21} = O.$$

As in 1), we may then assume that $l_{11} = O$ and $\det(h) \neq 0$, but now g may be singular. By induction, we have $a_1^{(n-r)}$, $b_1^{(n-r)}$, $c_1^{(n-r)}$ and $d_1^{(n-r)}$ such that

$$\overline{\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}} \begin{pmatrix} O & \bar{l}'_{22} \\ l_{22} & O \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}' = \begin{pmatrix} h_2^{(n-r)} & O \\ O & g_2^{(n-r)} \end{pmatrix},$$

$$\overline{\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}} \begin{pmatrix} O & I^{(n-r)} \\ -I^{(n-r)} & O \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}' = \begin{pmatrix} O & I^{(n-r)} \\ -I^{(n-r)} & O \end{pmatrix},$$

and we may assume that the rank of h_2 is higher than that of g_2, for otherwise

$$\begin{pmatrix} O & I \\ I & O \end{pmatrix}' \begin{pmatrix} h_2 & O \\ O & g_2 \end{pmatrix} \begin{pmatrix} O & I \\ I & O \end{pmatrix} = \begin{pmatrix} g_2 & O \\ O & h_2 \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} I & O \\ O & a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} O & O \\ O & b_1 \end{pmatrix}, \qquad C = \begin{pmatrix} O & O \\ O & c_1 \end{pmatrix}, \qquad D = \begin{pmatrix} I & O \\ O & d_1 \end{pmatrix}$$

then

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

belongs to $\mathfrak{G}$ and H_1 of $\bar{\mathfrak{T}} \mathfrak{H} \mathfrak{T}'$ is equal to $\begin{pmatrix} h & O \\ O & h_2 \end{pmatrix}$. Since its rank cannot be higher than r, we have $h_2 = O$. Consequently, $g_2 = O$. Then $l_{22} = O$. The result is now proved.

6. A lemma. For reasons explained in the Introduction, we find it necessary first to discuss the theory of pairs of Hermitian matrices **(6-9)**[2] as a basis for the classification of hypercircles **(10-16)**.

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[2] Cf. Dickson, *Modern algebraic theories*, p. 123, Theorem 10; MacDuffee, *Theory of matrices*, p. 63, Theorem 36. 5; Turnbull and Aitken, *Theory of canonical matrices*, p. 131, Lemma III; and Logsdon, *American Journal of Mathematics*, vol. 44 (1922), pp. 247-260. An earlier paper of Muth, *Journ. für Math.*, vol. 128 (1905), pp. 302-321 should be mentioned as one of importance in this connection.

Lemma. *If $q(x)$ is a polynomial, with real coefficients, which has no negative or zero root, then we have a real polynomial $\chi(x)$ such that $\chi^2(x)-x$ is divisible by $q(x)$.*

Proof. Let

$$q(x)=\lambda\prod_{i=1}^{s}(x-a_i)^{l_i}\prod_{j=1}^{t}((x-\alpha_i)(x-\bar{\alpha}_i)),$$

where $a_i>0$ and α_i is complex.

1) The theorem is true for

$$q(x)=(x-a)^l.$$

In fact the theorem is true for $l=1$, for then $\chi(x)=\sqrt{a}$ is a solution. Suppose that we have a real polynomial $\chi_{l-1}(x)$ such that

$$\chi^2_{l-1}(x)-x=(x-a)^{l-1}\lambda(x), \qquad l>1,$$

where $\lambda(x)$ is a polynomial with real coefficients. Evidently $\chi_{l-1}(a)\neq 0$. Then

$$\chi_l(x)=\chi_{l-1}(x)-\tfrac{1}{2}\frac{\chi(a)}{\chi_{l-1}(a)}(x-a)^{l-1}$$

satisfies our requirement, since

$$\begin{aligned}\chi^2_l(x)-x&\equiv\chi^2_{l-1}(x)-x-\frac{\lambda(a)}{\chi_{l-1}(a)}\chi_{l-1}(x)(x-a)^{l-1}\\&\equiv\left(\lambda(x)-\frac{\lambda(a)}{\chi_{l-1}(a)}\chi_{l-1}(x)\right)(x-a^{l-1})\\&\equiv 0 \pmod{(x-a)^l}.\end{aligned}$$

2) The theorem is true for

$$q(x)=((x-\alpha)(x-\bar{\alpha}))^l.$$

In fact

$$\chi(x)=\frac{1}{\sqrt{2\,|\,\alpha\,|+\alpha+\alpha}}(x+|\,\alpha\,|)$$

satisfies our requirement for $l=1$, since

$$\begin{aligned}\chi^2(x)-x&=\frac{1}{2\,|\,\alpha\,|+\alpha+\bar{\alpha}}(x^2-2\,|\,\alpha\,|\,x+|\,\alpha\,|^2)-x\\&=\frac{1}{2\,|\,\alpha\,|+\alpha+\bar{\alpha}}(x-\alpha)(x-\bar{\alpha})\\&\equiv 0 \pmod{(x-\alpha)(x-\bar{\alpha})}\end{aligned}$$

and $2\,|\,\alpha\,|+\alpha+\bar{\alpha}>0$.

Let $\chi_{l-1}(x)$ be a real polynomial satisfying

$$\chi^2_{l-1}(x) - x = ((x-\alpha)(x-\alpha))^{l-1}\lambda(x), \qquad l > 1.$$

It may be verified directly that

$$\chi_l(x) = \chi_{l-1}(x) + ((x-\alpha)(x-\bar{\alpha}))^{l-1}(sx+t)$$

satisfies our requirement, where the real numbers s and t are given by

$$\lambda(\alpha) + 2(s\alpha + t)\chi_{l-1}(\alpha) = 0$$

(The existence of s and t is easily seen, since α is not real and $\chi_{l-1}(\alpha) \neq 0$).

3) Let $q_1(x)$ and $q_2(x)$ be two real polynomials without common divisor, and let $\chi_1(x)$ and $\chi_2(x)$ be two real polynomials satisfying

$$\chi_1^2(x) - x \equiv 0 \pmod{q_1(x)}$$

and

$$\chi_2^2(x) - x \equiv 0 \pmod{q_2(x)}.$$

It is well-known that we have two real polynomials $h_1(x)$ and $h_2(x)$ such that

$$h_1(x)q_1(x) + h_2(x)q_2(x) = 1.$$

Then on letting

$$\chi(x) = \chi_1(x)h_2(x)q_2(x) + \chi_2(x)h_1(x)q_1(x),$$

we have

$$\chi^2(x) - x \equiv 0 \pmod{q_1(x)q_2(x)}.$$

Applying the process repeatedly, we have the theorem.

7. A theorem on pairs of Hermitian forms.

THEOREM 15. *If H and K are two Hermitian linear λ-matrices having the same elementary divisors, then we have two non-singular matrices Γ_1 and Γ_2 such that*

$$\begin{aligned} \bar{\Gamma}_1 H \Gamma'_1 &= h_1^{(r_1)} + h_2^{(r_2)}, \\ \bar{\Gamma}_2 K \Gamma'_2 &= k_1^{(r_1)} + k_2^{(r_2)}, \end{aligned} \qquad r_1 + r_2 = h, \quad r_1 \geq 0, \quad r_2 \geq 0\,[3]$$

and, we have two non-singular matrices $p_1^{(r_1)}$ and $p_2^{(r_2)}$ such that

$$\begin{aligned} \bar{p}_1 h_1 p'_1 &= k_1, \\ \bar{p}_2 h_2 p'_2 &= -k_2. \end{aligned}$$

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[3] In case $r_1 = 0$, $h^{(r_1)}$ is left out.

Proof. 1) By the hypothesis we have two non-singular matrices P and Q such that

$$PHQ = K.$$

Since $PHQ = PH\bar{p}' \cdot \bar{p}'^{-1}Q$, we may assume that $P = I$.

Since H and K are both Hermitian we have

$$HQ = \bar{Q}'H = K.$$

We have a non-singular matrix T such that

$$T^{-1}QT = q_1{}^{(r_1)} \dotplus q_2{}^{(r_2)}$$

where q_1 has non-negative characteristic roots and q_2 has only negative characteristic roots. We may assume without loss of generality that

$$Q = q_1 \dotplus q_2.$$

Let

$$H = \begin{pmatrix} h_{11} & h_{12} \\ \bar{h}'_{12} & h_{22} \end{pmatrix}.$$

Since $HQ = \bar{Q}'H$, we have $h_{12}q_2 = \bar{q}'_1 h_{12}$. Since $\bar{q}'_1, q_2$ have no common characteristic root, then $h_{12} = O$. Thus

$$H = h_1{}^{(r_1)} \dotplus h_2{}^{(r_2)}.$$

Consequently

$$K = k_1{}^{(r_1)} \dotplus k_2{}^{(r_2)},$$

and

$$h_1 q_1 = \bar{q}'_1 h_1 = k_1, \qquad h_2 q_2 = \bar{q}'_2 h_2 = k_2.$$

2) In the lemma of **6** we take $q(x)$ to be the characteristic polynomial of q_1. Then we have a real polynomial $\chi(x)$ such that

$$\chi^2(q_1) = q_1.$$

Then, letting $p_1 = \chi(q_1)$, we have

$$k_1 = h_1 q_1 = h_1 \chi^2(q_1) = \chi(\bar{q}'_1) h_1 \chi(q_1) = \bar{p}'_1 h_1 p_1.$$

Next, in the lemma of **6**, we take $q(x)$ to be the characteristic polynomial of $-q_2$. Then we have a polynomial $\chi(x)$ such that

$$\chi^2(-q_2) = -q_2.$$

Let $p_2 = \chi(-q_2)$, then

$$k_2 = h_2 q_2 = -h_2 \chi^2(-q_2) = -\chi(-\bar{q}'_2) h_2 \chi(-q_2) = -\bar{p}'_2 h_2 p_2.$$

The theorem is then proved.

8. Canonical form of pairs of Hermitian forms. First of all, we introduce the following notations: Let

$$j^{(t)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be a t-rowed square matrix (a_{ij}) with

$$a_{ij} = \begin{cases} 1 & \text{for } i + j = n + 1, \\ 0 & \text{otherwise;} \end{cases}$$

and let

$$m^{(t)}(\lambda) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \lambda \\ 0 & 0 & \cdots & \lambda & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \lambda & \cdots & 0 & 0 \\ \lambda & 1 & \cdots & 0 & 0 \end{pmatrix}$$

be a t-rowed square matrix (b_{ij}) with

$$b_{ij} = \begin{cases} \lambda & \text{for } i + j = n + 1, \\ 1 & \text{for } i + j = n + 2, \\ 0 & \text{otherwise.} \end{cases}$$

(In case $n = 1$, then $b_{11} = \lambda$).

THEOREM 16. *Let (A, B) and (A_1, B_1) be two pairs of Hermitian matrices. Let* $\det(\lambda A + B) = 0$ *have no real root and let A and A_1 be non-singular. A necessary and sufficient condition for the pairs to be conjunctive is that they have the same elementary divisors. More definitely, given*

$$g_i = ((\lambda - \lambda_1)(\lambda - \bar{\lambda}_1))^{t_{i1}} \cdots ((\lambda - \lambda_k)(\lambda - \bar{\lambda}_k))^{t_{ik}},$$

where $1 \leq i \leq n$ *and g_i divides g_{i+1} and $\Sigma t_{ij} = \frac{1}{2}n$. Let*

$$J = \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & j^{(t_{ij})} \\ j^{(t_{ij})} & O \end{pmatrix}$$

and

$$M = \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & m^{(t_{ij})}(\lambda_i) \\ m^{(t_{ij})}(\bar{\lambda}_i) & O \end{pmatrix}$$

where the Σ's denote direct sums and, for $t_{ij} = 0$, the corresponding term is to be left out. Then $\lambda J - M$ has the preassigned g_i as its i-th elementary

divisor. Further every pair of Hermitian matrices (A, B) with g_i as its i-th elementary divisor is conjunctive to (J, M).

Proof. It is not difficult to verify that $\lambda J - M$ has g_i as its i-th elementary divisor.

1) In Theorem 15, we take

$$H = \lambda A - B, \qquad K = \lambda J - M.$$

If $r_2 = 0$, the theorem is evident. If $r_1 = 0$, then we have a non-singular matrix P such that $\bar{P}HP' = -K$. Let

$$Q = \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} I^{(t_{ij})} & 0 \\ 0 & -I^{(t_{ij})} \end{pmatrix}.$$

Then $\bar{Q}JQ' = -J$ and $\bar{Q}MQ' = -M$. Thus $\bar{Q}\bar{P}HP'Q' = K$ and the theorem is true.

2) Consider first the particular case where we have

$$g_n = ((x - \alpha)(x - \bar{\alpha}))^{n/2}$$

and $g_{n-1} = \cdots = g_1 = 1$. $\lambda J - M$ cannot be conjunctive to a direct sum of two Hermitian matrices. For otherwise we would have two non-singular matrices P and Q such that

$$P(\lambda J - M)Q = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}$$

and p_1 and p_2 are Hermitian. Then either $(x - \alpha)^{n/2}$ or $(x - \bar{\alpha})^{n/2}$, and hence both, would divide $\det(p_1)$. This is impossible. Then we have either $r_1 = 0$ or $r_2 = 0$ in this case. The result is then true for this particular case.

3) If $r_1 \neq 0$, $r_2 \neq 0$, then we have to consider h_1 and h_2 in Theorem 15 separately. Applying induction on the number of the distinct invariant factors, we have the theorem.

THEOREM 17. *Every pair (A, B),* $\det(A) \neq 0$, *of Hermitian matrices is conjunctive to the following pair (J, M), where*

$$J = \dot{\sum_i} \dot{\sum_j} \epsilon_{ij} j^{(s_{ij})} \dotplus \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} 0 & j^{(t_{ij})} \\ j^{(t_{ij})} & 0 \end{pmatrix},$$

$$M = \dot{\sum_i} \dot{\sum_j} \epsilon_{ij} m^{(s_{ij})}(c_i) \dotplus \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} 0 & m^{(t_{ij})}(\lambda_i) \\ m^{(t_{ij})}(\bar{\lambda}_i) & 0 \end{pmatrix};$$

the first $\dot{\sum_i}$ *runs over all real roots of* $\det(\lambda A + B) = 0$ *and the second* $\dot{\sum_i}$ *runs over all pairs of complex roots of* $\det(\lambda A + B) = 0$, *and* $\epsilon_{ij} = \pm 1$.

The proof of this theorem is completely analogous to that of Theorem 16.

DEFINITION. *The pair of forms* (J, M) *obtained in Theorem* 17 *is called the canonical form of all the pairs conjunctive to it.*

For a fixed c, we may arrange s_{ij} as

$$\begin{aligned} s_{i1} = s_{i2} = \cdots = s_{i\alpha} &> s_{i\alpha+1} = \cdots = s_{i\alpha+\beta} \\ &> s_{i\alpha+\beta+1} = \cdots = s_{i\alpha+\beta+\gamma} \\ &> \cdots = \cdots = s_{i\alpha+\beta+\cdots+\eta}. \end{aligned}$$

We set

$$\begin{aligned} \sigma_1{}^{(i)} &= \epsilon_{i1} + \cdots + \epsilon_{i\alpha}, \\ \sigma_2{}^{(i)} &= \epsilon_{i\alpha+1} + \cdots + \epsilon_{i\alpha+\beta}, \\ \sigma_3{}^{(i)} &= \epsilon_{i\alpha+\beta+1} + \cdots + \epsilon_{i\alpha+\beta+\gamma}. \end{aligned}$$

The constants $\sigma_1{}^{(i)}, \sigma_2{}^{(i)}, \cdots$ are called *the system of signatures of the pairs of forms with respect to the real root c.*

To each real root we have a system of signatures. The totality of all the elementary divisors and all the systems of signatures is called the system of elementary divisors with signatures.

9. Law of inertia.

THEOREM 18. *The system of elementary divisors with signatures characterize the conjunctivity of pairs of Hermitian matrices completely. More exactly, the elementary divisors and the systems of signatures are the same for all conjunctive pairs of Hermitian matrices (law of inertia); pairs with different elementary divisors or with the same elementary divisors but different systems of signatures are not conjunctive.*

Proof. 1) It is known that if two pairs of Hermitian matrices are conjunctive, then their elementary divisors are the same. Further, it is evident that two canonical pairs with the same elementary divisors and the same system of signatures are conjunctive.

Thus it is sufficient to establish the result by showing that any two canonical pairs of Hermitian matrices with the same elementary divisors but different systems of signatures are not conjunctive.

2) Let (J, M) and

$$J_1 = \dot{\sum_i} \sum_j \epsilon'_{ij} j^{(s_{ij})} \dot{+} \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & j^{(t_{ij})} \\ j^{(t_{ij})} & O \end{pmatrix}$$

$$M_1 = \dot{\sum_i} \sum_j \epsilon_{ij} m^{(s_{ij})}(c_i) \dot{+} \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & m^{(t_{ij})}(\lambda_i) \\ m^{(t_{ij})}(\bar{\lambda}_i) & O \end{pmatrix}$$

be two canonical pairs of Hermitian matrices with the same elementary divisors. If (J, M) and (J_1, M_1) are conjunctive, then we have a non-singular $n \times n$ matrix Γ such that

$$\bar{\Gamma}(J, M)\Gamma' = (J_1, M_1).$$

Then

$$\bar{\Gamma}(MJ^{-1}) = (MJ^{-1})\Gamma,$$

since $MJ^{-1} = M_1 J_1^{-1} = \bar{\Gamma}(MJ^{-1})\bar{\Gamma}^{-1}$. Since $J^2 = I$, and

$$MJ^{-1} = \dot{\sum_i} \dot{\sum_j} m^{(s_{ij})}(c_i) j^{(s_{ij})} \dot{+} \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & m^{(t_{ij})}(\lambda_i) j^{(t_{ij})} \\ m^{(t_{ij})}(\bar{\lambda}_i) j^{(t_{ij})} & O \end{pmatrix}$$

we have

$$\Gamma = \dot{\sum_i} \Gamma_i \dot{+} \dot{\sum_i} \begin{pmatrix} \Gamma_{11}^{(i)} & \Gamma_{12}^{(i)} \\ \Gamma_{21}^{(i)} & \Gamma_{22}^{(i)} \end{pmatrix}$$

and

$$\bar{\Gamma}_i (\dot{\sum_j} m^{(s_{ij})}(c_i) j^{(s_{ij})}) = (\dot{\sum_j} m^{(s_{ij})}(c_i) j^{(s_{ij})}) \bar{\Gamma}_i.$$

Also

$$\bar{\Gamma}_i (\dot{\sum} \epsilon_{ij} j^{(s_{ij})}) \Gamma'_i = \dot{\sum} \epsilon'_{ij} j^{(s_{ij})}$$

$$\bar{\Gamma}_i (\dot{\sum} \epsilon_{ij} m^{(s_{ij})}(c_i)) \Gamma'_i = \dot{\sum} \epsilon'_{ij} m^{(s_{ij})}(c_i).$$

Thus it is sufficient to prove the theorem for the case with a unique real root c.

3) We require a

LEMMA. *Let H^A denote the adjoint matrix of H.*

(i) *If H and K are two conjunctive non-singular Hermitian λ-matrices, then H^A and K^A are conjunctive also; furthermore, if we arrange H^A and K^A as polynomials in λ, then their corresponding coefficients (which are matrices) are conjunctive.*

(ii) *If* $\det(H) \neq 0$ *and*

$$H = h_1 \dot{+} h_2 \dot{+} \cdots \dot{+} h_t,$$

then

(iii)

$$\frac{H^A}{d(H)} = \frac{h_1{}^A}{d(h_1)} \dot{+} \frac{h_2{}^A}{d(h_2)} \dot{+} \cdots \dot{+} \frac{h_t{}^A}{d(h_t)} .$$

$$(m^{(t)}(\lambda))^A = (-1)^{\frac{1}{2}(t-1)(t-2)} \begin{pmatrix} 1, & -\lambda, & \lambda^2, \cdots, & (-\lambda)^{t-1} \\ -\lambda, & \lambda^2, & -\lambda^3, \cdots, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ (-\lambda)^{t-1}, & 0, & 0, \cdots, & 0 \end{pmatrix}$$

which is a t-rowed square matrix (a_{ij}) *with*

$$a_{ij} = \begin{cases} (-\lambda)^{i+j+2} & \text{for } i+j \leq t+1, \\ 0 & \text{otherwise.} \end{cases}$$

All these results may be verified easily.

4) Since

$$\bar{\Gamma}(J, M)\Gamma' = (J_1, M_1),$$

we have

$$\bar{\Gamma}((\lambda - c)J + M)\Gamma' = (\lambda - c)J_1 + M_1$$

for any λ. We write, dropping the subscript i,

$$M(\lambda) = (\lambda - c)J + M = \dot{\sum_j} \epsilon_j m^{(s_j)}(\lambda)$$

and

$$M_1(\lambda) = (\lambda - c)J_1 + M_1 = \dot{\sum_j} \epsilon'_j m^{(s_j)}(\lambda).$$

They are conjunctive for any λ. Thus $\det(M(\lambda))$ and $\det(M_1(\lambda))$ have the same sign, i. e., $\prod_j \epsilon_j{}^{s_j} = \sum_j \epsilon'_j{}^{s_j}$, since

$$\det(\epsilon_j m^{(s_j)}(\lambda)) = (-1)^{\frac{1}{2}s_j(s_j-1)}(\epsilon_j \lambda)^{s_j}.$$

Further, let

$$\Pi\, \epsilon_j{}^{s_j}(-1)^{\frac{1}{2}s_j(s_j-1)} = \epsilon,$$

$$M(\lambda)^A = \epsilon\lambda^n \left(\dot{\sum_j} \frac{\epsilon_j (m^{(s_j)}(\lambda))^A}{\det(m^{(s_j)}(\lambda))} \right)$$
$$= \epsilon \dot{\sum_j} \epsilon_j (-1)^{\frac{1}{2}s_j(s_j-1)} (m^{(s_j)}(\lambda))^A \lambda^{n-s_j}.$$

The coefficient of λ^{n-s_1} is equal to

$$\epsilon(-1)^{(s_1-1)} \dot{\sum_{1 \leq j \leq a}} \epsilon_j \begin{pmatrix} 1 & 0 & 0 \cdots & 0 \\ 0 & 0 & 0 \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \cdots & 0 \end{pmatrix},$$

since $(-1)^{\frac{1}{2}s_1(s_1-1)}(-1)^{\frac{1}{2}(s_1-1)(s_1-2)}=(-1)^{(s_1-1)}$. By (i) of the lemma, the signature of this matrix is equal to that of the corresponding expression of $M_1(\lambda)$; hence

$$\sum_{j=1}^{a}\epsilon_j=\sum_{j=1}^{a}\epsilon'_j.$$

The coefficient of $\lambda^{n-s_{a+1}}$ is of the form

$$\epsilon(\dot{\sum_{1\leq j\leq a}}\epsilon_j P_j)\dot{+}\epsilon(-1)^{(s_{a+1}-1)}\dot{\sum_{a+1\leq j\leq a+\beta}}\epsilon_j\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}.$$

The corresponding expression of $M_1(\lambda)$ may be written as

$$\epsilon(\dot{\sum_{1\leq j\leq a}}\epsilon_j P_j)\dot{+}\epsilon(-1)^{(s_{a+1}-1)}\dot{\sum_{a+1\leq j\leq a+\beta}}\epsilon'_j\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}$$

(by arranging the first part such that $\epsilon_i=\epsilon'_i$ for $1\leq i\leq \alpha$). Thus we have

$$\sum_{j=a+1}^{a+\beta}\epsilon_j=\sum_{j=a+1}^{a+\beta}\epsilon'_j.$$

The result follows by induction.

10. Normal form of hypercircles.

Theorem 19. *Every hypercircle is conjunctive under $\mathfrak{G}$ to a hypercircle with the matrix*

$$\begin{pmatrix}H_1 & O\\ O & H_2\end{pmatrix},\qquad H_1=\begin{pmatrix}h_1^{(r)} & O\\ O & O\end{pmatrix},\qquad H_2=\begin{pmatrix}h_2^{(r)} & O\\ O & O\end{pmatrix}$$

where h_1 and h_2 may be expressed as two direct sums

$$h_1=\dot{\sum_i}\dot{\sum_j}\epsilon_{ij}j^{(s_{ij})}\dot{+}\dot{\sum_i}\dot{\sum_j}\begin{pmatrix}O & j^{(t_{ij})}\\ j^{(t_{ij})} & O\end{pmatrix}$$

and

$$h_2=\dot{\sum_i}\dot{\sum_j}\epsilon_{ij}m^{(s_{ij})}(c_i)\dot{+}\dot{\sum_i}\dot{\sum_j}\begin{pmatrix}O & m^{(t_{ij})}(\lambda_i)\\ m^{(t_{ij})}(\bar{\lambda}_i) & O\end{pmatrix}$$

where the c's are real and the λ's are complex numbers.

Proof. By Theorem 14, we have only to consider the case with

$$H_1=\begin{pmatrix} h_1 & O \\ O & O \end{pmatrix}, \quad H_2=\begin{pmatrix} h_2 & O \\ O & O \end{pmatrix}, \quad \det(h_1)\neq 0.$$

Consider the pair of Hermitian matrices $(h_1^{-1}, \bar{h}_2)$.

By Theorem 17, we have a non-singular matrix γ such that

$$\bar{\gamma} h_1^{-1}\gamma' = \dot{\sum}\dot{\sum} \epsilon_{ij} j^{(s_{ij})} + \dot{\sum_i}\dot{\sum_j}\begin{pmatrix} O & j^{(t_{ij})} \\ j^{(t_{ij})} & O \end{pmatrix}$$

$$\bar{\gamma}\bar{h}_2\gamma' = \dot{\sum}\dot{\sum} \epsilon_{ij} m^{(s_{ij})}(c_i) + \dot{\sum_i}\dot{\sum_j}\begin{pmatrix} O & m^{(t_{ij})}(\bar{\lambda}_i) \\ m^{(t_{ij})}(\lambda_i) & O \end{pmatrix}.$$

Let

$$A=\begin{pmatrix} \bar{\gamma}'^{-1} & O \\ O & O \end{pmatrix}, \quad D=\begin{pmatrix} \bar{\gamma} & O \\ O & 1 \end{pmatrix}, \quad B=C=O, \quad \mathfrak{T}=\begin{pmatrix} A & B \\ C & D \end{pmatrix};$$

$\mathfrak{T}$ belongs to $\mathfrak{G}$. Then

$$\bar{\mathfrak{T}}\begin{bmatrix} \begin{pmatrix} h_1 & O \\ O & O \end{pmatrix} & \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & O \\ O & O \end{pmatrix} & \begin{pmatrix} h_2 & O \\ O & O \end{pmatrix} \end{bmatrix}\mathfrak{T}'$$

gives the required form. (Notice that

$$\Bigl(\dot{\sum_{i,j}}\dot{\sum} \epsilon_{ij} j^{(s_{ij})} + \dot{\sum_i}\dot{\sum_j}\begin{pmatrix} O & j^{(t_{ij})} \\ j^{(t_{ij})} & O \end{pmatrix}\Bigr)^2 = I\Bigr).$$

THEOREM 20. *Every hypercircle with a matrix of the form given in Theorem 19 has a canonical discriminantal matrix. Apart from ϵ_{ij}, all other quantities in the expression of the matrix of the hypercircle are completely determined by its discriminantal matrix.*

The proof of the theorem needs only a direct verification.

Thus for a given discriminantal matrix we have only *a finite number* of hypercircles, more exactly, the number of hypercircles is $\leqq 2n$. We have to consider further whether the forms given in Theorem 19 are equivalent. The answer will be given in **15.**

11. Complete reducibility.

DEFINITION. *A sub-set $\mathfrak{C}$ of $\mathfrak{G}$ is said to be completely reducible, if we have a transformation $\mathfrak{W}$ belonging to $\mathfrak{G}$ such that the elements of $\mathfrak{W}^{-1}\mathfrak{C}\mathfrak{W}$ are of the form*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$A = \begin{pmatrix} a_1 & O \\ O & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & O \\ O & b_2 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & O \\ O & c_2 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & O \\ O & d_2 \end{pmatrix}.$$

THEOREM 21. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be two hypercircles with the same discriminantal matrix $\mathfrak{D}$, and let* det $(\mathfrak{D} - \lambda\mathfrak{F}) = 0$ *have more than one distinct root. The transformations which carry $\mathfrak{H}$ to $\mathfrak{K}$ are completely reducible. In particular, if $\mathfrak{H} = \mathfrak{K}$, they form a completely reducible group.*

Proof. We may assume that

$$\mathfrak{D} = \begin{pmatrix} O & T \\ -T' & O \end{pmatrix},$$

where $T = t_1 \dotplus t_2$ and t_1 and t_2 have no common characteristic roots.

Suppose that $\bar{\mathfrak{T}}\mathfrak{H}\mathfrak{T}' = \mathfrak{K}$ where $\mathfrak{T}$ belongs to $\mathfrak{G}'$, then $\mathfrak{T}\mathfrak{D}\mathfrak{T}' = \mathfrak{D}$. Since $\mathfrak{T}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}$ we have $\mathfrak{T}'^{-1} = -\mathfrak{F}\mathfrak{T}\mathfrak{F}$. Then $\mathfrak{T}\mathfrak{D} = -\mathfrak{D}\mathfrak{F}\mathfrak{T}\mathfrak{F}$.
Put

$$\mathfrak{F} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} O & T \\ -T' & O \end{pmatrix} = \begin{pmatrix} O & T \\ -T' & O \end{pmatrix}\begin{pmatrix} D & -C \\ -B & A \end{pmatrix}$$

i. e.,

$$T'C = CT, \qquad T'D = DT',$$
$$TA = AT, \qquad TB = BT'.$$

Since t_1, t_2 have no common characteristic root, we have

$$A = a_1 \dotplus a_2, \qquad B = b_1 \dotplus b_2,$$
$$C = c_1 \dotplus c_2, \qquad D = d_1 \dotplus d_2.$$

The theorem follows.

In order to investigate the conjunctivity under $\mathfrak{G}$ of the forms in Theorem 19, we need only investigate the conjunctivity under $\mathfrak{G}$ of

$$h_1 = \dot{\sum} \epsilon_i j^{(t_i)}$$
$$h_2 = \dot{\sum} \epsilon_i m^{(t_i)}(c)$$

where c is a real number. The solutions are quite different according to $c < 0$, > 0 or $= 0$.

12. Conjunctivity under $\mathfrak{G}$ for $c > 0$.

THEOREM 22. *The hypercircle with the matrix*

$$\begin{pmatrix} j^{(t)} & O \\ O & m^{(t)}(c) \end{pmatrix}$$

is conjunctive under $\mathfrak{G}$ to that with

$$-\begin{pmatrix} j^{(t)} & O \\ O & m^{(t)}(c) \end{pmatrix}$$

provided $c < 0$.

Proof. We shall first establish the following preliminary result:

We have a real and symmetric matrix $s^{(t)}$ such that

$$sj^{(t)}s = -m(c),$$

if $c < 0$.

The result is true for $t = 1$, since

$$\sqrt{|c|} \cdot 1 \cdot \sqrt{|c|} = -c, \text{ i. e., } s = \sqrt{-c}.$$

The result is also true for $t = 2$, since

$$\begin{pmatrix} 0 & \sqrt{-c} \\ \sqrt{-c}, & -\tfrac{1}{2}(\sqrt{-c})^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{-c} \\ \sqrt{-c}, & -\tfrac{1}{2}(\sqrt{-c})^{-1} \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & c \\ c & 1 \end{pmatrix}, \quad \text{i. e.,} \quad s = \begin{pmatrix} 0 & \sqrt{-c} \\ \sqrt{-c}, & -\tfrac{1}{2}(\sqrt{-c})^{-1} \end{pmatrix}.$$

Suppose that the theorem is true for t, then we shall prove that it is also true for $t + 2$, i. e., suppose we have s such that

$$sjs = -m(c)$$

and

$$\det(sj + \sqrt{-c}\, I^{(t)}) \neq 0.$$

Then, we solve

$$\begin{pmatrix} 0 & 0 & z \\ 0 & s^{(t)} & w' \\ z & w & u \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & j^{(t)} & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & z \\ 0 & s & w' \\ s & w & u \end{pmatrix} = -m^{(t+2)}(c)$$

i. e., we find real numbers z, u and a t-dimensional vector w such that

$$z^2 = -c, \qquad w'z + sjw' = \begin{pmatrix} -1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \qquad 2uz + wjw' = 0.$$

The first equation gives $z = \sqrt{-c}$, the second is then soluble in w if and only if

$$\det (sj + \sqrt{-c}\, I^{(t)}) \neq 0$$

which is true by assumption, and from the third we then have the value of u. Set

$$s^{(t+2)} = \begin{pmatrix} 0 & 0 & z \\ 0 & s^{(t)} & w' \\ s & w & u \end{pmatrix}$$

where z, w, u are determined in this way; then $s^{(t+2)}$ satisfies

$$s^{(t+2)} j^{(t+2)} s^{(t+2)} = -m^{(t+2)}(c)$$

and

$$\begin{aligned} &\det (s^{(t+2)} j^{(t+2)} + \sqrt{-c}\, I^{(t+2)}) \\ &= -4c \det (s^{(t)} j^{(t)} + \sqrt{-c}\, I^{(t)}) \neq 0. \end{aligned}$$

The preliminary result is now proved. Let

$$\mathfrak{T} = \begin{pmatrix} O & s^{-1} \\ -s & O \end{pmatrix}$$

which belongs to $\mathfrak{G}$. Then

$$\bar{\mathfrak{T}} \begin{pmatrix} j^{(t)} & O \\ O & m^{(t)}(c) \end{pmatrix} \mathfrak{T}' = \begin{pmatrix} s^{-1} m(c) s^{-1} & O \\ O & sjs \end{pmatrix} = -\begin{pmatrix} j^{(t)} & O \\ O & m(c) \end{pmatrix}.$$

The theorem follows.

Consequently, the signs ϵ_{ij} corresponding to a negative c_i in Theorem 19 may be replaced by $+1$.

13. Conjunctivity under $\mathfrak{G}$ for $c > 0$.

THEOREM 23. *If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are conjunctive under $\mathfrak{G}$, then the two pairs of Hermitian matrices*

$$(\bar{\mathfrak{H}}_1, \mathfrak{F}\mathfrak{H}_1{}^A\mathfrak{F})$$

and

$$(\bar{\mathfrak{H}}_2, \mathfrak{F}\mathfrak{H}_2{}^A\mathfrak{F})$$

are also conjunctive under $\mathfrak{G}$.

Proof. Let $\mathfrak{T}$ be an element of $\mathfrak{G}$ and $\bar{\mathfrak{T}}\mathfrak{H}_1\mathfrak{T}' = \mathfrak{H}_2$. Since $\mathfrak{T}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}$ and $\mathfrak{T}^{-1} = \mathfrak{T}^A$ we have

$$\mathfrak{H}_2{}^A = \mathfrak{T}'^A\mathfrak{H}_1{}^A\mathfrak{T}^A = \mathfrak{T}'^{-1}\mathfrak{H}_1{}^A\mathfrak{T}^{-1}$$

and

$$\mathfrak{F}\mathfrak{H}_2{}^A\mathfrak{F} = \mathfrak{F}\mathfrak{T}'^{-1}\mathfrak{H}_1{}^A\mathfrak{T}^{-1}\mathfrak{F} = \mathfrak{T}\mathfrak{F}\mathfrak{H}_1{}^A\mathfrak{F}\mathfrak{T}'.$$

Therefore

$$\mathfrak{T}(\lambda\bar{\mathfrak{H}}_1 + \mu\mathfrak{F}\mathfrak{H}_1{}^A\mathfrak{F})\bar{\mathfrak{T}}' = \lambda\bar{\mathfrak{H}}_2 + \mu\mathfrak{F}\mathfrak{H}_2{}^A\mathfrak{F}.$$

THEOREM 24. *Let $c > 0$, and*

$$h_1 = \dot{\sum} \epsilon_i j^{(s_i)}$$
$$h_2 = \dot{\sum} \epsilon_i m^{(s_i)}(c).$$

For different systems of signatures we have non-conjunctive hypercircles (under $\mathfrak{G}$) with matrices

$$\mathfrak{H} = \begin{pmatrix} h_1 & O \\ O & h_2 \end{pmatrix}$$

under $\mathfrak{G}$.

Proof. Let

$$\mathfrak{H} = \begin{pmatrix} h_1 & O \\ O & h_2 \end{pmatrix}, \quad \mathfrak{K} = \begin{pmatrix} k_1 & O \\ O & k_2 \end{pmatrix}$$

be two such hypercircles with different systems of signatures. If they are conjunctive under $\mathfrak{G}$, then

$$(\mathfrak{H}, -\mathfrak{F}\mathfrak{H}^A\mathfrak{F}) = \left(\begin{pmatrix} h_1 & O \\ O & h_2 \end{pmatrix}, \begin{pmatrix} \det(h_1)h_2{}^A & O \\ O & \det(h_2)h_1{}^A \end{pmatrix}\right)$$

and

$$(\mathfrak{K}, -\mathfrak{F}\mathfrak{K}^A\mathfrak{F}) = \left(\begin{pmatrix} k_1 & O \\ O & k_2 \end{pmatrix}, \begin{pmatrix} \det(k_1)k_2{}^A & O \\ O & \det(k_2)k_1{}^A \end{pmatrix}\right)$$

are conjunctive.

We shall now prove that

$$\phi = \lambda h_1 + \mu \det(h_1)h_2{}^A$$

is conjunctive to

$$\psi = \lambda h_2 + \mu \det(h_2) h_1^A.$$

We have

$$h_1^A \phi h_2 = h_1^A(\lambda h_1 + \mu \det(h_1) h_2^A) h_2$$
$$= \det(h_1)(\lambda h_2 + \mu \det(h_2) h_1^A) = \det(h_1)\psi.$$

Then

$$h_1^A \phi \bar{h}_1'^A \cdot h_1 h_2 = (\det(h_1))^2 \psi.$$

Now $[\det(h_1)]^2$ is positive and $h_1 h_2$ is a matrix with a positive characteristic root c. Hence as in the proof of Theorem 15, we have a matrix p such that

$$p h_1^A \phi \bar{h}_1'^A \bar{p}' = \psi.$$

Thus ϕ and ψ have the same system of elementary divisors with the same systems of signatures. Thus if $(\mathfrak{H}, \mathfrak{F}\mathfrak{H}^A\mathfrak{F})$ and $(\mathfrak{K}, \mathfrak{F}\mathfrak{K}^A\mathfrak{F})$ are conjunctive, then

$$(h_1, \det(h_2) h_1^A), \qquad (k_2, \det(k_2) k_1^A$$

are conjunctive, then (since $h_1^{-1} = h_1$, $k_1^{-1} = k_1$),

$$(h_2, h_1), \qquad (k_2, k_1)$$

are conjunctive. By Theorem 16, they are conjunctive if and only if they have the same systems of signatures.

Consequently the signs ϵ_{ij} corresponding to a positive c_i in Theorem 19 are significant.

14. Conjunctivity under $\mathfrak{G}$ for $c = 0$.

Here we require a preliminary lemma.

LEMMA. Let

$$t^{(l)} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

be an l-rowed matrix. The solution of

$$x^{(l,m)} t^{(m)} = t^{(l)} x^{(l,m)}$$

is of the form

$$x^{(l,m)} = \begin{pmatrix} x_1, & x_2, & \cdots, & x_m \\ 0, & x_1, & \cdots, & x_{m-1} \\ \cdot & \cdot & \cdots & \cdot \\ 0, & 0, & \cdots, & x_1 \\ 0, & 0, & \cdots, & 0 \\ \cdot & \cdot & \cdots & \cdot \end{pmatrix} \quad \textit{if } l > m,$$

$$x^{(l,m)} = \begin{pmatrix} 0 & \cdots & 0 & x_1, & x_2, & \cdots, & x_l \\ 0 & \cdots & 0 & 0, & x_1, & \cdots, & x_{l-1} \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 0, & 0, & 0, & \cdots, & x_1 \end{pmatrix} \quad \textit{if } l > m,$$

$$x^{(l,l)} = \begin{pmatrix} x_1, & x_2, & \cdots, & x_l \\ 0, & x_1, & \cdots, & x_{l-1} \\ \cdot & \cdot & \cdots & \cdot \\ 0, & 0, & \cdots, & x_1 \end{pmatrix}.$$

THEOREM 25. *Theorem 24 is also true for* $c = 0$.

Proof. Let

$$\mathfrak{H} = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}, \qquad \mathfrak{K} = \begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix},$$

and let

$$T = H'_1 H_2 = K'_1 K_2 = \sum_i j^{(s_i)} m^{(s_i)}(0),$$

where

$$j^{(s)} m^{(s)}(0) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

(For $s = 1$, it is zero.)

Evidently $H_1{}^2 = K_1{}^2 = I$. Let

$$\bar{\mathfrak{T}} \mathfrak{H} \mathfrak{T}' = \mathfrak{K}, \qquad \mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Since $\mathfrak{T}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}$, $\mathfrak{F}^2 = -I^{(2n)}$ and $\mathfrak{T}\mathfrak{D}\mathfrak{T}' = \mathfrak{D}$, we have $\mathfrak{T}\mathfrak{D}\mathfrak{F} = \mathfrak{D}\mathfrak{F}\mathfrak{T}$. Now $\mathfrak{D} = \begin{pmatrix} O & T \\ -T' & O \end{pmatrix}$; consequently, we have

$$AT = TA, \qquad BT' = TB, \qquad CT = T'C, \qquad DT' = T'D.$$

Now we use Greek letters to denote matrices commutative with T. Then

$$A=\alpha, \qquad B=\beta H_1, \qquad C=H_1\gamma, \qquad D=H_1\delta H_1,$$

since $T'=H_1TH_1$. Since

$$\overline{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}\begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}'=\begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix},$$

we have

$$K_1=\bar{A}H_1A'+\bar{B}H_2B'=\bar{\alpha}H_1\alpha'+\bar{\beta}H_1H_2H_1\beta'=\bar{\alpha}H_1\alpha'+T\bar{\beta}H_1\beta'.$$

Write

$$K_1=(k_{ij})_{1\leq i,j\leq\kappa}$$

with

$$k_{ii}=\epsilon'_{ij}{}^{(s_i)}, \qquad k_{ij}=0 \quad \text{for} \quad i\neq j.$$

Similarly, we write

$$H_1=(h_{ij})_{1\leq i,j\leq\kappa}$$

with

$$h_{ii}=\epsilon_{ij}{}^{(s_i)}, \qquad h_{ij}=0 \quad \text{for} \quad i\neq j.$$

Further, we write

$$T=(t_{ij})$$

with

$$t_{ij}=j^{(s_i)}m^{(s_i)}(0), \qquad t_{ij}=0 \quad \text{for} \quad i\neq j;$$

and finally, we write

$$\alpha=(a_{ij}), \qquad a_{ij}=a_{ij}{}^{(s_i,s_j)}.$$

Then

$$k_{ij}=\sum_{\lambda,\mu}\bar{a}_{i\lambda}h_{\lambda\mu}a'_{j\mu}+\sum_{\lambda\ldots}t_{i\lambda}\cdots.$$

Now we consider the element in the $(s_i,1)$-position. The contribution from k_{ij} is either ϵ'_i for $i=j$ or 0 for $i\neq j$. The contribution from $\sum_{\lambda\ldots} t_{i\lambda}\cdots$ is zero, since the last row of $t_{i\lambda}$ is zero.

By the lemma, since

$$a_{ik}t_{kk}=t_{ii}a_{ik},$$

we have

$$a_{ik}=\begin{pmatrix} 0 & \cdots & 0, & x_{ik} & * & \cdots & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ 0, & \cdot & \cdot & \cdot & \cdot & 0 & x_{ik} \end{pmatrix} \quad \text{for } s_i>s_k,$$

$$\text{or} = \begin{pmatrix} x_{ik} & * & \cdots & * \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & x_{ik} \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \text{for } s_i < s_k,$$

$$\text{or} = \begin{pmatrix} x_{ik} & * & \cdots & * \\ 0 & x_{ik} & \cdots & * \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & x_{ik} \end{pmatrix} \quad \text{for } s_i = s_k.$$

The element in the $(s_i, 1)$-position of $\bar{a}_{i\lambda}h_{\lambda\mu}a'_{j\mu}$ is zero for $\lambda \neq \mu$; is zero for $s_i < s_\lambda$; is zero for $s_j > s_\mu$; and is

$$\sum_{s_\lambda = s_i = s_j} \bar{x}_{i\lambda}\epsilon_\lambda x_{j\lambda} \quad \text{for} \quad s_\lambda = s_\mu = s_i = s_j.$$

Thus we obtain

$$\sum_{s_\lambda = s_i = s_j} \bar{x}_{i\lambda}\epsilon_\lambda x_{j\lambda} = \begin{cases} \epsilon'_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let all the elements s_ν equal to s_μ be

$$s_{\eta+1}, \cdots, s_{\eta+\xi}.$$

Then

$$\begin{pmatrix} \epsilon'_{\eta+1} & 0 & \cdots & 0 \\ 0 & \epsilon'_{\eta+2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \epsilon'_{\eta+\xi} \end{pmatrix} = (\overline{x_{ij}}) \begin{pmatrix} \epsilon_{\eta+1} & 0 & \cdots & 0 \\ 0 & \epsilon_{\eta+2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \epsilon_{\eta+\xi} \end{pmatrix} (x_{ij})'.$$

Thus

$$\epsilon_{\eta+1} + \cdots + \epsilon_{\eta+\xi} = \epsilon'_{\eta+1} + \cdots + \epsilon'_{\eta+\xi}.$$

The result follows.

15. Canonical form of hypercircles. We now summarize the results of **10-14**.

THEOREM 26. *Every hypercircle is conjunctive under $\mathfrak{G}$ to a hypercircle with the matrix*

$$\begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}, \quad H_1 = \begin{pmatrix} h_1^{(r)} & O \\ O & O \end{pmatrix}, \quad H_2 = \begin{pmatrix} h_2^{(r)} & O \\ O & O \end{pmatrix},$$

where h_1 and h_2 may be expressed as two direct sums

$$h_2 = \dot{\sum_{c_i \geqq 0}} \dot{\sum} \epsilon_{ij} m^{(s_{ij})}(c_i) \dot{+} \dot{\sum_{c_i < 0}} \dot{\sum} m^{(s_{ij})}(c_i) \dot{+} \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & m^{(t_{ij})}(\lambda_i) \\ m^{(t_{ij})}(\lambda_i) & O \end{pmatrix}$$

and

$$h_1 = \dot{\sum_{c_i \geqq 0}} \dot{\sum} \cdot \epsilon_{ij} j^{(s_{ij})} \dot{+} \dot{\sum_{c_i < 0}} \dot{\sum} j^{(s_{ij})} \dot{+} \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & j^{(t_{ij})} \\ j^{(t_{ij})} & O \end{pmatrix},$$

where the first double summation runs over non-negative c's, the second runs over negative c's and the third runs over all complex λ's.

Moreover, to each non-negative c, we may define the system of signatures as we did for the pairs of Hermitian matrices. Elementary divisors and systems of signatures characterize completely the conjunctivity of hypercircles under $\mathfrak{G}$.

Thus the problem of the conjunctivity of hypercircles under $\mathfrak{G}$ is now solved completely.

16. A final remark.

The treatment is much simpler for the case of the group $\mathfrak{G}_{II}$ which consists of all transformations of the form

$$Z_1 = (AZ + B)(CZ + D)^{-1},$$
$$A\bar{B}' = B\bar{A}', \qquad C\bar{D}' = D\bar{C}' \qquad A\bar{D}' - B\bar{C}' = I.$$

It is evident that a transformation with the matrix

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

belongs to $\mathfrak{G}_{II}$ if and only if

$$\bar{\mathfrak{T}} \mathfrak{F} \mathfrak{T}' = \mathfrak{F}.$$

Correspondingly, the transformation of hypercircles may be written as

$$\bar{\mathfrak{T}} \mathfrak{H} \mathfrak{T}' = \mathfrak{K}.$$

Thus, the pair $\mathfrak{H}, \mathfrak{K}$ are conjunctive under $\mathfrak{G}_{II}$ in the strict sense, if and only if the pairs of Hermitian matrices

$$(\mathfrak{H}, i\mathfrak{F}), \qquad (\mathfrak{K}, i\mathfrak{F})$$

are conjunctive.

The classification of the hypercircles under $\mathfrak{G}_{II}$ is thus simply a straightforward application of the preceding results on pairs of Hermitian forms.

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DIOPHANTINE APPROXIMATIONS AND HILBERT'S SPACE.*

By Aurel Wintner.

1. The object of this paper is a class of *complete* sequences of functions which, in view of its connections with various problems of the analytic theory of numbers, is a class of arithmetical significance. It is understood that a sequence of functions $f_1(t), f_2(t), \cdots$, where $a \leqq t \leqq b$, is said to be complete if it forms a basis of the realization $L^2(a, b)$ of Hilbert's space (that is, if the sequence can be orthogonalized into one satisfying Parseval's relation for every function of class (L^2) on the interval $a \leqq t \leqq b$). The complete sequences in question result from an appropriate form of an analytical counterpart of the sieve of Eratosthenes. This is illustrated by the following result on the fundamental function of the theory of Diophantine approximations:

(I) *If* (t) *denotes the least non-negative residue of* t (mod 1),

$$(1) \qquad (t) = t - [t],$$

then the sequence

$$(2) \qquad 1, (t), (2t), \cdots, (nt), \cdots$$

is (L^2)*-complete on the interval* $0 \leqq t \leqq \frac{1}{2}$.

As will be seen from the proof, the numerical value, $\frac{1}{2}$, of the length of the interval is introduced by the formal circumstance that the Fourier expansion of the underlying periodic function (1), or rather of the first Bernoullian function

$$(3) \qquad B_1(t) \equiv t - [t] - \tfrac{1}{2} \sim -\sum_{k=1}^{\infty} (\pi k)^{-1} \sin 2\pi k t; \qquad 0 \leqq t \leqq 1,$$

contains only the sequence of the *odd* harmonics of the interval $0 \leqq t \leqq 1$; a sequence which is complete only for $0 \leqq t \leqq \frac{1}{2}$. The adjunction of a non-vanishing constant, 1, to the functions (nt) in (2) is necessitated by the fact that the mean-value of (3), but not of (1), is 0 over a period. Actually, there results a sequence which is *not* complete on the Hilbert space $L^2(0, \frac{1}{2})$, if *any* of the terms of the sequence (2) is omitted. In other words, if

$$(4) \qquad c_0.1 + c_1.(t) + c_2.(2t) + \cdots + c_n.(nt) = 0, \qquad (0 \leqq t \leqq \tfrac{1}{2})$$

* Received November 17, 1943.

(almost everywhere), then every c is 0. This is readily seen from the geometrical structure of the graph representing the successive transforms of the broken linear function (1). Another proof results if the Fourier series of the functions $(t), (2t), \cdots, (nt)$ (having the respective periods $1, \frac{1}{2}, \cdots, 1/n$) are substituted into (4), since the uniqueness theorem of Fourier series (of period $1/n!$) then asserts that $c_1 = 0, \cdots, c_n = 0$ and $c_0 + c_1 + \cdots + c_n = 0$.

The formal transition from the completeness of the system

$$(5) \qquad 1, \phi(t), \phi(2t), \cdots, \phi(nt), \cdots$$

$(0 \leqq t \leqq \frac{1}{2})$, where $\phi(t) = \sin 2\pi t$, to the completeness of the system (2), where $\phi(t) = B_1(t)$, depends on the fact that the sieve-process of Eratosthenes is reversible. It will be convenient to use this reversibility in its explicit form, as expressed by Möbius' inversion formula. In fact, the latter implies that, if either of two sequences, say $A(1), \cdots, A(n), \cdots$ and $B(1), \cdots, B(n), \cdots$, is given arbitrarily, then the other sequence is uniquely determined by the assignment

$$(6) \qquad B(n) = \sum_{d|n} A(d), \qquad (n = 1, 2, \cdots);$$

the explicit inversion of the linear transformation (6) of the sequence $A(1), A(2), \cdots$ into the sequence $B(1), B(2), \cdots$ being

$$(7) \qquad A(n) = \sum_{d|n} \mu(n/d) B(d), \qquad (n = 1, 2, \cdots)$$

(μ is Möbius' factor, defined by the identity

$$(8) \qquad \prod_p (1 - p^{-s}) = \sum_{n=1}^{\infty} \mu(n) n^{-s}, \qquad (s > 1),$$

where p runs through all primes). It is understood that the summation index, d, in (5) and (6) runs through all positive divisors of n (including $d = 1$ and $d = n$).

2. The italicized assertion (I) with regard to the sequence (2) can be thought of as a dual of the approximation theorem of Kronecker-Weyl (or, rather, Jacobi-Bohl), according to which the sequence (2) is of uniform asymptotic distribution on the interval $(0, 1)$ for every fixed irrational t (and so, in particular, for almost every fixed t). In addition, if the sequence (2) is interpreted as representing the successive rotations of a circumference by a fixed irrational angle, then, according to G. D. Birkhoff and P. A. Smith, the resulting measure-preserving transformation of the interval $(0, 1)$ is

metrically transitive (but it is not a mixture). Thus there arises the question as to a description of further duals of the ergodic theorem which correspond to the assertion (I) in cases in which the irrational rotations are replaced by *less explicit* measure-preserving transformations of $(0,1)$. But I did not succeed in obtaining a satisfactory characterization of these peculiar ergodic transformations, supplying, as in the particular case (2), a linear basis of the (L^2)-space carried by a suitable subset of the set $(0,1)$.

On the other hand, generalizations of the completeness property of (2) in an arithmetical, rather than a measure-theoretical, direction can be proved without more effort than in the case (2) itself. In particular, the assertion of **1** concerning (3) will be generalized as follows:

(II) *If the index* λ *(which may be complex) is such as to make either of the trigonometric series*

$$(9_1)\qquad \phi(t)\sim\sum_{k=1}^{\infty}k^{-\lambda}\cos 2\pi kt;\qquad\qquad (9_2)\qquad \phi(t)\sim\sum_{k=1}^{\infty}k^{-\lambda}\sin 2\pi kt$$

a Fourier series (L^2), *i. e., if*

$$(10)\qquad \Re\lambda>\tfrac{1}{2},$$

then the sequence (5) *belonging to either of the functions* (9_1), (9_2) *of period* 1 *is* (L^2)*-complete on the interval* $0\leqq t\leqq \frac{1}{2}$.

This includes not only (3) but also the higher Bernoullian functions $B_2(t), B_3(t),\cdots$ and their extensions to a fractional index (with the limitation (10) of the index). Actually, the explicit arithmetical structure of the coefficients in (9_1), (9_2) will be fully needed only in order to make available Möbius' inversion itself, rather than the generalizations of (6), (7) to the case of an arbitrary "Dirichlet inversion" (in this regard, cf. Hölder [5], Landau [6] and **5-6** below).

3. The proofs will be based on a combination of Möbius' inversion with a treatment which I applied some time ago to a trigonometric series considered by Riemann (cf. Wintner [9] and, for generalizations, Hartman and Wintner [3], Hartman [2]), as follows:

(III) Let an integrable function of period 1 and of mean-value 0, say the function

$$(11)\qquad \phi(t)\sim\sum_{k=1}^{\infty}(a_k\cos 2\pi kt+b_k\sin 2\pi kt),\qquad (a_0=0),$$

satisfy the (L^2)-condition

$$(12)\qquad \sum_{k=1}^{\infty} (|\, a_k\,|^2 + |\, b_k\,|^2) < \infty$$

because of the existence of an $\epsilon > 0$ for which

$$(13)\qquad a_k = O(k^{-\frac{1}{2}-\epsilon}), \qquad b_k = O(k^{-\frac{1}{2}-\epsilon}),$$

and let $c_1, c_2, \cdots$ be any sequence satisfying

$$(14)\qquad \sum_{n=1}^{\infty} |\, c_n\,|^2 < \infty$$

because of the existence of an $\eta > 0$ for which

$$(15)\qquad c_n = O(n^{-\frac{1}{2}-\eta}).$$

Then the series

$$(16)\qquad \sum_{n=1}^{\infty} c_n \phi(nt)$$

is convergent in the mean of (L^2) on $(0,1)$. Furthermore, if $f(t)$ denotes the function of class $L^2(0,1)$ to which the partial sums

$$(17)\qquad f_n(t) = c_1\phi(t) + c_2\phi(2t) + \cdots + c_n\phi(nt)$$

of (16) converge in the mean (L^2), then the Fourier analysis of $f(t)$ is given by

$$(18)\qquad f(t) \sim \sum_{k=1}^{\infty} (\alpha_k \cos 2\pi kt + \beta_k \sin 2\pi kt), \qquad (\alpha_0 = 0),$$

where, if $d(\geqq 1)$ runs through all divisors of k,

$$(19)\qquad \alpha_k = \sum_{d|k} c_d\, a_{k/d}, \qquad \beta_k = \sum_{d|k} c_d\, b_{k/d}.$$

In fact, from (11),

$$(20)\qquad \phi(nt) \sim \sum_{k=1}^{\infty} (a_k \cos 2\pi nkt + b_k \sin 2\pi nkt)$$

for $n = 1, 2, \cdots$. Hence, from (17),

$$(21)\qquad f_n(t) \sim \sum_{k=1}^{\infty} (\alpha_k^n \cos 2\pi kt + \beta_k^n \sin 2\pi kt),$$

if the coefficients $\alpha_1^n, \beta_1^n, \alpha_2^n, \beta_2^n, \cdots$ belonging to a fixed n are defined by

$$(22) \qquad \alpha_k{}^n = \sum_{d|k}^{d\leqq n} c_d\, a_{k/d}, \qquad \beta_k{}^n = \sum_{d|k}^{d\leqq n} c_d\, b_{k/d}.$$

If n is replaced by m both in (21) and (22), and if $m > n$, it follows, by applying Parseval's relation to the function $f_m(t) - f_n(t)$, that

$$\int_0^1 |\, f_m(t) - f_n(t)\,|^2\, dt = \sum_{k=n+1}^{\infty} \Big(\Big| \sum_{d|k}^{n<d\leqq m} c_d a_{k/d} \Big|^2 + \Big| \sum_{d|k}^{n<d\leqq m} c_d b_{k/d} \Big|^2 \Big).$$

Accordingly,

$$(23) \qquad \int_0^1 |\, f_m(t) - f_n(t)\,|^2\, dt \leqq \sum_{k=n+1}^{\infty} (A_k{}^2 + B_k{}^2) \quad \text{for} \quad m > n,$$

if A_k, B_k are abbreviations for the finite sums

$$(24) \qquad A_k = \sum_{d|k} |\, c_d a_{k/d}\,|, \qquad B_k = \sum_{d|k} |\, c_d b_{k/d}\,|.$$

Consequently, it is sufficient to ascertain the existence of a sufficiently small $\delta > 0$ satisfying

$$(25) \qquad A_k = O(k^{-\frac{1}{2}-\delta}), \qquad B_k = O(k^{-\frac{1}{2}-\delta}).$$

In fact, if (25) is assured, then the inequality (23) implies that the integral on the left of (23) tends to 0 as $n \to \infty$, $m \to \infty$. But the compactness of Hilbert's space then supplies the existence of a function $f(t)$, of class (L^2), satisfying

$$(26) \qquad \int_0^1 |\, f(t) - f_n(t)\,|^2\, dt \to 0 \quad \text{as} \quad n \to \infty;$$

so that

$$(27) \qquad f(t) \sim \sum_{n=1}^{\infty} c_n \phi(nt),$$

if (27) is meant to signify that the partial sums, (16), of (17) tend to $f(t)$ in the mean of (L^2) on $(0, 1)$. Finally, if α_k, β_k denote, as in (18), the Fourier constants of $f(t)$, then the assertion (19), where $\alpha_0 = 0$, follows from (21), (22) and (26), since, if k is fixed, the finite sums (22) tend obviously to the corresponding finite sums (19), as $n \to \infty$.

The truth of (25) is implied by those assumptions which have not been used thus far, namely, by (13) and (15). In fact, it is seen from these assumptions and from the definitions (24), that both A_k and B_k are majorized by a fixed multiple of

$$\sum_{d|k} d^{-\frac{1}{2}-\eta} (k/d)^{-\frac{1}{2}-\epsilon} = k^{-\frac{1}{2}-\epsilon} \sum_{d|k} d^{\epsilon-\eta}.$$

Hence, if $\epsilon > 0$ and $\eta > 0$ in (13) and (15) are so chosen that $\epsilon = 2\eta$, then both A_k and B_k are majorized by a constant multiple of

$$k^{-\frac{1}{2}-\epsilon} \sum_{d|k} d^{-\epsilon} = k^{-\frac{1}{2}-\epsilon} \sum_{d|k} (k/d)^{-\epsilon} = k^{-\frac{1}{2}-2\epsilon} \sum_{d|k} d^{\epsilon}.$$

But it is easily seen that the sum of the ϵ-th powers of all divisors of k is $O(k^{\epsilon+\omega})$, if $\epsilon > 0$ and $\omega > 0$ are arbitrarily fixed (for logarithmic refinements, which express the best possible asymptotic inequalities but which are not needed here, cf. Gronwall [1]). Hence, if ω is chosen to be $\frac{1}{2}\epsilon$, the last of the above estimates of A_k and B_k shows that (25) is satisfied by $\delta = \frac{1}{2}\epsilon$.

This completes the proof of the existence of (18), (27) and of the representation (19) of the Fourier inversion.

4. The assumption (13) is an explicit refinement of the (L^2)-assumption (12) for (11), and (15) is a corresponding refinement of (14). It is clear from the above proof for the existence of the function (27), that (13) and (15) can be improved logarithmically, but that the proof fails if only (13) and (15) are assumed. One might think that this failure is due to the method alone. But it turns out that such is not the case. In other words, *the existence of a function* (27) *is not an issue in Hilbert's space, since* (12) *and* (14) *do not imply that the partial sums of the series* (16) *defined by* (11) *converge in the mean* (L^2); *not even if* (12) *is refined to* (13).

In fact, even if (11) is chosen to be the function (3) belonging to the sequence (2) of the assertion of **1** (a function which, being of bounded variation, satisfies (13) for $\epsilon = \frac{1}{2}$), the replacement of (15) by (14) will still lead to series (16) for which the function to be defined by (27) does not exist. This follows from (IV) below, since the Dirichlet series corresponding to the odd Fourier series (3) is

$$\sum_{k=1}^{\infty} (\pi k)^{-1} k^{-s} = \pi^{-1} \sum_{k=1}^{\infty} k^{-s-1} = \zeta(s+1)/\pi \tag{28}$$

and has therefore a pole at $s = 1$. The general criterion runs as follows:

(IV) *For a given function* $\phi(t)$ *of period* 1 *and of class* (L^2), *defined by* (11) *and* (12), *the partial sums of the periodic series* (16) *belonging to an arbitrary sequence* $c_1, c_2, \cdots$ *satisfying* (14) *cannot converge in the mean* (L^2) *unless the Fourier constants of* $\phi(t)$ *are such that both functions defined by the Dirichlet series*

$$(29_1) \quad \sum_{k=1}^{\infty} a_k k^{-s}; \qquad (29_2) \quad \sum_{k=1}^{\infty} b_k k^{-s}$$

are regular-analytic and bounded in the half-plane $\Re s > 0$.

6

All that (12) ensures, by Schwarz's inequality, is the absolute convergence of (29_1) and (29_2) in the half-plane $\Re s > \frac{1}{2}$; in fact, if $a_n = 0$ and $b_n = n^{-\frac{1}{2}} \log(n+1)$, then (12) is satisfied but (29_2) diverges at $s = \frac{1}{2}$ and is, in addition, not a bounded function in the half-plane $\Re s > \frac{1}{2}$. But the deficiency of the (L^2)-condition (12) is even greater than the possible divergence of (29_1), (29_2) in the critical strip $0 < \Re s < \frac{1}{2}$; in fact, the example (3), (28) shows that not even the convergence of (29_1), (29_2) on the whole of this critical strip can guarantee that the condition required by (IV) is satisfied. What is true is the converse, namely the fact that (29_1), (29_2) must converge on the whole of the critical strip *if* the functions represented by (29_1), (29_2) in the half-plane $\Re s > \frac{1}{2}$ possess analytic continuations which remain regular and bounded in the half-plane $\Re s > 0$ (Bohr); a converse quite irrelevant for the present problem.

The proof of (IV) will be based on a connection between the sieve of Eratosthenes and the D-matrices of Toeplitz [7]; a connection pointed out, but not further followed, before (Wintner [8]). This connection is obscured by the fact that, in contrast with the Eratosthenian algorithm underlying (19) and (22), Toeplitz's representation of his bilinear forms is not such as to correspond to an infinite matrix of *zeilenfinit* type. But the arithmetical significance of this discrepancy is merely the Eratosthenian counterpart of the difference between Riemann's and Lebesgue's ways of reading the integral of a continuous function.

It may be mentioned that, if only (12) is assumed for the function (11), then, for any pair of positive integers n, m,

$$\int_0^1 \phi(nt)\phi(mt)\,dt = \tfrac{1}{2}\sum_{k=1}^{\infty} \left(a_{kn/(n,m)}a_{km/(n,m)} + b_{kn/(n,m)}b_{km/(n,m)}\right),$$

where (n, m) denotes the greatest common divisor of n and m (so that the subscripts of a and b run through two fixed multiples of the summation index k). This is easily seen from (20), if the polarized form of Parseval's relation is applied to the pair $\phi(nt), \phi(mt)$.

5. Inasmuch as (13) and (15) have been used only when proving that the integral on the left of (23) tends to 0 as $n \to \infty$, $m \to \infty$, it is clear from the arrangement of the proof in **3** that, if only (12) and (14) are assumed, and if $\phi(t)$ and $f_n(t)$ are defined by (11) and (21) respectively, then there exists a function $f(t)$, of class (L^2), satisfying (26) and having the Fourier expansion (18) in which the coefficients are given by (19). This implies that, if a_k, b_k, c_k are arbitrary constants satisfying (12) and (14), then their bilinear combinations (19) must satisfy

$$\sum_{k=1}^{\infty} (|\alpha_k|^2 + |\beta_k|^2) < \infty, \tag{30}$$

by Parseval's relation. Hence, if either the numbers a_k, c_k, α_k or the numbers b_k, c_k, β_k are denoted, respectively, by λ_k, x_k, y_k, it is clear that (IV) is contained in the second part of the following criterion:

(V) *If $\lambda_1, \lambda_2, \cdots$ is a fixed sequence of constants, the vector $(y_1, y_2, \cdots)$, into which an arbitrary vector $(x_1, x_2, \cdots)$ is transformed by the corresponding Eratosthenian substitution*

$$y_k = \sum_{d|k} \lambda_{k/d} x_d, \qquad (k = 1, 2, \cdots), \tag{31}$$

becomes a point $(y_1, y_2, \cdots)$ of Hilbert's space for every point $(x_1, x_2, \cdots)$ of Hilbert's space if and only if the Dirichlet series

$$\sum_{k=1}^{\infty} \lambda_k k^{-s} \tag{32}$$

defines a function regular and bounded in the half-plane $\Re s > 0$.

According to a fundamental criterion of Hellinger and Toeplitz [4], an infinite matrix is bounded (in Hilbert's sense) if and only if it transforms every point of Hilbert's space into a point of Hilbert's space. Hence, if (31) is thought of as written in the form

$$y_k = \sum_{l=1}^{\infty} \mu_{kl} x_l, \qquad (k = 1, 2, \cdots), \tag{33}$$

(so that, for instance, $\mu_{kl} = 0$ whenever $l > k$), then the assertion of (V) is equivalent to the statement that the infinite matrix (μ_{kl}) is bounded if and only if the Dirichlet series (32) defines a function regular and bounded in the half-plane $\Re s > 0$.

Toeplitz [7] has shown that (32) defines such a function if and only if the infinite "D-matrix" which he associated with the coefficients $\lambda_1, \lambda_2, \cdots$ of (32) is bounded. Consequently, the assertion of (V) is that the D-matrix belonging to (32) is bounded if and only if the matrix (μ_{kl}) is. But the D-matrix belonging to (32) is defined as follows: The first row consists of the sequence $\lambda_1, \lambda_2, \cdots, \lambda_l, \cdots$ (λ_l being the l-th element of the first row), and the k-th row results if one inserts $k-1$ zeros in front of every element of the sequence $\lambda_1, \lambda_2, \cdots$. Hence, it is readily realized that the *transposed* matrix of this D-matrix is identical with the matrix (μ_{kl}) which is obtained if (31) is written in the form (33). In fact, the identity of the matrix (μ_{kl})

with the *transposed* D-matrix is just a concise formulation of the sieve-process of Eratosthenes.

Since a matrix is bounded if and only if its transposed matrix is, the proof of (V) is complete.

6. The proof of (II) proceeds as follows:

Let (11) be given either by (9_1) or by (9_2); so that

(34_1) $\quad a_k = k^{-\lambda},\ b_k = 0;$ $\qquad\qquad$ (34_2) $\quad a_k = 0,\ b_k = k^{-\lambda}$

in the respective cases. Then, since (10) is assumed, (25) is satisfied. Hence, if $c_1, c_2, \cdots$ is any sequence satisfying (15), the functions (17) are such that there will exist a function $f(t)$, of class (L^2), satisfying (27). But this $f(t)$ will then be given by (18) and (19). And (34_1), (34_2) show that (19) can now be written in the form

$$(35_1) \qquad k^\lambda \alpha_k = \sum_{d|k} d^\lambda c_d, \qquad \beta_k = 0;$$

$$(35_2) \qquad \alpha_k = 0, \qquad k^\lambda \beta_k = \sum_{d|k} d^\lambda c_d$$

in the respective cases (9_1), (9_2).

Consequently, in order to prove (II), it would be sufficient to show that, if $f(t)$ is any given function of class (L^2) on the interval $0 \leqq t \leqq \frac{1}{2}$, and if $\alpha_1, \alpha_2, \cdots$ in (35_1) or $\beta_1, \beta_2, \cdots$ in (35_2) are *given* by either of the Fourier series

$$(36_1) \qquad f(t) - \tfrac{1}{2}\alpha_0 \sim \sum_{k=1}^{\infty} \alpha_k \cos 2\pi k t, \qquad (0 \leqq t \leqq \tfrac{1}{2}),$$

$$(36_2) \qquad f(t) \sim \sum_{k=1}^{\infty} \beta_k \sin 2\pi k t, \qquad (0 \leqq t \leqq \tfrac{1}{2}),$$

(so that all that is known of the *given* constants α_k or β_k is

(37_1) $\quad \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty;$ $\qquad\qquad$ (37_2) $\quad \sum_{k=1}^{\infty} |\beta_k|^2 < \infty$

in the respective cases), then the constants $c_1, c_2, \cdots$ assigned by either of the infinite systems (35_1), (35_2) of linear equations must satisfy (15). But it will be shown in **7** that this plan cannot be carried out, i. e., that (37_j) *and* (35_j) *do not imply* (15), where either $j = 1$ or $j = 2$. However, the plan needs only a slight modification.

In fact, if $j = 1$, it is sufficient to carry out the plan for each of the *particular* functions $f(t) = \cos 2\pi k t$, where k is a fixed positive integer. For

suppose that (35_1) and (36_1) are verified to imply (15) when (36_1) is given by a fixed $f(t) = \cos 2\pi kt$. Then (26) and (17) show that this particular $f(t)$ is within the (L^2)-range of the linear function-space generated by the basis

$$(38) \qquad \phi(t), \phi(2t), \cdots, \phi(nt), \cdots.$$

On the other hand, all these particular functions $f(t)$ and the constant 1, that is, the functions

$$(39_1) \qquad 1, \cos 2\pi t, \cdots, \cos 2\pi kt, \cdots,$$

form a linear basis of the whole (L^2)-space on $0 \leqq t \leqq \frac{1}{2}$. Since (5) results by adjoining to (38) the first element of (39_1), the assertion of (II) for the case $j = 1$ now follows by adding two $\frac{1}{2}\epsilon$'s, the (L^2)-space on $0 \leqq t \leqq \frac{1}{2}$ being a metric space.

If $j = 2$, then (39_1) becomes replaced by

$$(39_2) \qquad \sin 2\pi t, \cdots, \sin 2\pi kt, \cdots.$$

However, the transition from (38) to (5) is still needed, as illustrated by the transition from (3) to (1) and (2) in the particular case (I) of (II).

What remains to be shown is that (35_j) and (36_j) imply (15) when $f(t) = \cos 2\pi nt$ or $f(t) = \sin 2\pi nt$ according as $j = 1$ or $j = 2$, where n is any fixed positive integer in either case. For reasons of symmetry, it will be sufficient to consider the case $j = 2$.

Clearly, (35_2) can be written in the form (6) by placing

$$(40_2) \qquad A(k) = k^\lambda c_k, \qquad B(k) = k^\lambda \beta_k.$$

Since the linear transformation (6) of $A(1), A(2), \cdots$ into $B(1), B(2), \cdots$ has the unique inverse (7), it follows that (35_2) is equivalent to

$$(41_2) \qquad k^\lambda c_k = \sum_{d|k} \mu(k/d) d^\lambda \beta_d.$$

Suppose now that (36_2) is given by $f(t) = \sin 2\pi nt$, where n is fixed. This means that $\beta_k = e_{kn}$, where (e_{kn}) is the unit matrix. Thus (41_2) shows that

$$(42_{2n}) \qquad c_k = k^{-\lambda} \mu(k/n) n^\lambda \quad \text{if} \quad n \mid k, \quad \text{and} \quad c_k = 0 \text{ otherwise.}$$

Since (8) implies that $|\mu| \leqq 1$, it follows that $|c_k| \leqq |k^{-\lambda} n^\lambda|$ for every k; so that, since n is fixed, $c_k = O(|k^{-\lambda}|)$ as $k \to \infty$. Hence, (15) is assured by (10).

This completes the proof of (II), and therefore that of (I).

7. In order to prove the truth of the negation italicized after (37_2), it is more than sufficient to show that (37_2) *and* (35_2) *do not imply* (14).

It turns out that (37_2) and (35_2) fail to imply (14) even in the case $\lambda = 1$ of (3) or (1), where (13) is satisfied by $\epsilon = \frac{1}{2}$. In fact, (35_2) is equivalent to (41_2), which means, if $\lambda = 1$, that

$$c_k = \sum_{d|k} \mu(k/d)(k/d)^{-1}\beta_d. \tag{43}$$

Hence, the assertion is that (14) does not follow from (37_2), if c_k is defined by (43). But (43) can be written in the form (31), where

$$\lambda_k = \mu(k)k^{-1} \quad \text{and} \quad x_k = \beta_k,\ y_k = c_k. \tag{44}$$

It follows therefore from (V) that the assertion is equivalent to the statement that the Dirichlet series (32) does not define a function regular and bounded in the half-plane $\Re s > 0$, if $\lambda_k = \mu(k)k^{-1}$. But (8) shows that this Dirichlet series is identical with $1/\zeta(s+1)$, since the product (8) is the reciprocal value of Euler's product for $\zeta(s)$.

Since $\zeta(s)$ is known to become arbitrarily close to 0 in the half-plane $\Re s > 1$, it follows that the function (32) is not bounded in the half-plane $\Re s > 0$; so that the proof is complete.

8. The method applied in **6** proves not only (II) but certain variants of (II) as well. In this direction, the following analogue of (I) seems to be of particular interest:

(VI) *If* $\phi(t)$ *is of period* 1 *and* $\phi(t) = t/|t| \equiv \pm 1$ *for* $0 < \pm t < \frac{1}{2}$, *then the sequence* (5) *is* (L^2)*-complete on the interval* $0 \leqq t \leqq \frac{1}{4}$.

First, in the same way as both (39_1) and (39_2) are (L^2)-complete on the interval $0 \leqq t \leqq \frac{1}{2}$, the sequence

$$1,\quad \psi(t),\quad \psi(3t),\quad \psi(5t), \cdots \tag{39*}$$

is (L^2)-complete on the interval $0 \leqq t \leqq \frac{1}{4}$ if either $\psi(t) = \cos 2\pi t$ or $\psi(t) = \sin 2\pi t$. Hence, the proof of (II) in **6** can obviously be transcribed to the case where k in (9_1), (9_2) is restricted to be odd, if the interval $0 \leqq t \leqq \frac{1}{2}$ occurring in the assertion of (II) is replaced by $0 \leqq t \leqq \frac{1}{4}$; so that (II) has the following variant:

(VII) *If* λ *satisfies* (10), *and if* $\phi(t)$ *denotes either of the functions*

$$\phi(t) \sim \sum_{k=1}^{\infty} (2k-1)^{-\lambda} \cos 2\pi(2k-1)t; \tag{9_1*}$$

$$(9_2^*) \qquad \phi(t) \sim \sum_{k=1}^{\infty} (2k-1)^{-\lambda} \sin 2\pi(2k-1)t,$$

then the sequence (5) *is* (L^2)-*complete on the interval* $0 \leqq t \leqq \frac{1}{4}$.

This contains (VI), since $4/\pi$ times the series (9_2^*) belonging to $\lambda = 1$ is the Fourier series of the function $\phi(t)$ defined in (VI).

9. The above considerations contain a general criterion for the boundedness of certain non-negative definite Hermitean matrices, derived from an arbitrary function, of class (L^2), by the sieve of Eratosthenes, as follows:

(VIII) *Let* $\phi(t)$ *be a real-valued function which is of class* (L^2), *has the period* 1 *and the mean-value*

$$(45) \qquad \int_0^1 \phi(t)\,dt = 0,$$

and is either even or odd. In either case, the infinite matrix

$$(46) \qquad \left(\int_0^1 \phi(kt)\phi(lt)\,dt\right), \qquad (k, l = 1, 2, \cdots)$$

is bounded if and only if the Dirichlet series

$$(47) \qquad \sum_{k=1}^{\infty} \lambda_k k^{-s}$$

defines a function regular and bounded in the half-plane $\Re s > 0$, *where* $\lambda_1, \lambda_2, \cdots$ *denote the real Fourier constants of* $\phi(t)$, *as given by*

$$(48_1) \qquad \phi(t) \sim \sum_{k=1}^{\infty} \lambda_k \cos 2\pi kt; \qquad (48_2) \qquad \phi(t) \sim \sum_{k=1}^{\infty} \lambda_k \sin 2\pi kt$$

in the respective cases (so that

$$(49) \qquad \sum_{k=1}^{\infty} \lambda_k^2 < \infty$$

in either case).

The non-negative character of the (real) symmetric matrix (46) is clear from the identity

$$(50) \qquad \sum_{k=1}^{n} \sum_{l=1}^{n} g_{kl} c_k c_l = \int_0^1 f_n(t)^2\,dt \geqq 0,$$

where (g_{kl}) and $f_n(t)$ denote the matrix (46) and the function (17) respectively. This identity might suggest for matrices of the structure of (46)

a rule corresponding to Toeplitz's description of the spectra of his L-matrices. However, the determination of the spectrum involves delicate questions of "interference" in the case of (46).

In view of (V), the assertion of (VIII) is equivalent to the statement that (46) is bounded if and only if (μ_{kl}) is bounded, where (μ_{kl}) denotes, as in **5**, the matrix which results if (31) is written in the form (33). But a real matrix, say M, is bounded if and only if the matrix product $M'M$, where M' denotes the transposed matrix, exists and is bounded (Hellinger and Toeplitz [4]). If this fact is applied to $M=(\mu_{kl})$, it follows that (VIII) will be proved if it is verified that the boundedness of the matrix (46) is equivalent to the (existence and) boundedness of the matrix (G_{kl}), where G_{kl} denotes the series

$$\sum_{h=1}^{\infty}\mu_{hk}\mu_{hl}. \tag{51}$$

Since (12) and (11) are now represented by (49) and (48_j), where either $j=1$ or $j=2$, the bilinear identity of the last formula line in **4** is applicable and can be written in the form

$$\int_0^1 \phi(kt)\phi(lt)\,dt=\tfrac{1}{2}\sum_{h=1}^{\infty}\lambda_{hk/(k,l)}\lambda_{hl/(k,l)} \tag{52}$$

(in either case), where (k,l) denotes the greatest common divisor of k and l. Correspondingly, the balance of the proof of (VIII) follows from the sieve of Eratosthenes. In fact, since (μ_{kl}) denotes the matrix which results if (31) is written in the form (33), the series (51) turns out to be identical with the series multiplying $\frac{1}{2}$ on the right of (52).

Needless to say, the existence of the matrix formed by the elements (51), i. e., the convergence of the series (51) or (52), is assured by (49), as seen from Schwarz's inequality.

10. The criterion (VIII) becomes of particular arithmetical interest in case the Fourier constants of (48_1), (48_2) are "completely multiplicative" functions of their index, i. e., if

$$\lambda_r=\lambda_m\lambda_n \quad \text{when} \quad r=mn, \tag{53}$$

where $\lambda_1\neq 0$ (hence $\lambda_1=1$).

In this case, the series (49) possesses the Eulerian factorization

$$\sum_{k=1}^{\infty}\lambda_k^2=\prod_p\left(1+\sum_{l=1}^{\infty}\lambda_p^{2l}\right), \tag{54}$$

where p runs through all primes; so that (49) is equivalent to

$$(55) \qquad \sum_p \lambda_p{}^2 < \infty.$$

It is also clear from (53) that the h-th term of the series multiplying $\frac{1}{2}$ in (52) is identical with $\lambda_h{}^2 \lambda_{(k;l)}$, if $(k;l)$ is an abbreviation for the positive integer

$$(56) \qquad (k;l) = kl/(k,l)^2.$$

Thus (52) can be written in the form

$$(57) \qquad \int_0^1 \phi(kt)\phi(lt)dt = \text{const.}\, \lambda_{(k;l)}; \qquad \text{const.} = \tfrac{1}{2}\Pi,$$

where Π denotes the positive number (54). Finally, (47) admits, by (53), the Eulerian factorization

$$(58) \qquad \sum_{k=1}^{\infty} \lambda_k k^{-s} = \prod_p (1 + (p^s - 1)^{-1}\lambda_p)$$

(at least for $\Re s > \frac{1}{2}$).

Accordingly, (VIII) implies the following boundedness criterion of arithmetical nature:

(IX) *For any sequence of real numbers* $\lambda_1, \lambda_2, \cdots$ *satisfying* (53) *and* (55), *the infinite matrix*

$$(59) \qquad (\lambda_{(k;l)}), \qquad (k, l = 1, 2, \cdots),$$

where $(k;l) = (l;k)$ *denotes the integer* (56), *is a bounded matrix if and only if the Eulerian product* (58) *defines a function regular and bounded in the half-plane* $\Re s > 0$.

An example of (53) is $\lambda_k = \lambda(k)k^{-\sigma}$, where $\lambda(k)$ denotes Liouville's coefficient and σ is any real constant. It turns out that (IX) then leads to exactly the same condition for σ as in the simplest possible case of (53), namely in the case $\lambda_k = k^{-\sigma}$.

If $\lambda_k = k^{-\sigma}$, where σ is a fixed real number, the series (58) represents the function $\zeta(s + \sigma)$, and is therefore regular and bounded in the half-plane $\Re s > 0$ if and only if the value of σ (which is *not* the real part of s) exceeds 1. It follows therefore from (IX) and (56) that *the infinite matrix*

$$(60) \qquad \left(\frac{(k,l)^{2\sigma}}{k^{\sigma} l^{\sigma}}\right), \qquad (k, l = 1, 2, \cdots),$$

is bounded if and only if $\sigma > 1$, where (k, l) denotes the greatest common divisor of k and l.

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REFERENCES.

[1] T. H. Gronwall, "Some asymptotic expressions in the theory of numbers," *Transactions of the American Mathematical Society*, vol. 14 (1913), pp. 113-122.

[2] P. Hartman, "On a class of arithmetical Fourier series," *American Journal of Mathematics*, vol. 60 (1938), pp. 66-74.

[3] P. Hartman and A. Wintner, "On certain Fourier series involving sums of divisors," *Travaux de l'Institut Mathématique de Tbilissi*, vol. 3 (1938), pp. 113-118.

[4] E. Hellinger and O. Toeplitz, "Grundlagen für eine Theorie der unendlichen Matrizen," *Mathematische Annalen*, vol. 69 (1910), pp. 289-330.

[5] O. Hölder, "Ueber gewisse der Möbiusschen Funktion $\mu(n)$ verwandte zahlentheoretische Funktionen, die Dirichletsche Miltiplikation und eine Verallgemeinerung der Umkehrungsformeln," *Berichte der Sächs. Akad. Wiss., Math.-phys. Kl.*, vol. 85 (1933), pp. 1-28.

[6] E. Landau, "Ueber den Wertevorrat von $\zeta(s)$ in der Halbebene $\sigma > 1$," *Göttinger Nachrichten*, (1933), pp. 81-91.

[7] O. Toeplitz, "Zur Theorie der Dirichletschen Reihen," *American Journal of Mathematics*, vol. 60 (1938), pp. 880-888.

[8] A. Wintner, "Ueber die Spektra der Toeplitzschen D-Formen," *Monatshefte für Mathematik und Physik*, vol. 48 (1939), pp. 147-152.

[9] A. Wintner, "On a trigonometrical series of Riemann," *American Journal of Mathematics*, vol. 59 (1937), pp. 629-634.

A SUMMATION METHOD ASSOCIATED WITH DIRICHLET'S DIVISOR PROBLEM.*

By Aurel Wintner.

1. For any function $s = s_n$ of the positive integer n, let $D_n(s)$ denote the function

$$D_n(s) = \sum_{m=1}^{n} (n/m - [n/m]) s_m/n \tag{1}$$

of n, where $[x]$ is the greatest integer not exceeding x. This linear transformation of $s_1, s_2, \cdots$, when multiplied by the positive constant $(1 - C)^{-1}$, where C is the Euler-Mascheroni number, has the same structure of weighted averaging as the transformation of $s_1, s_2, \cdots$ into the successive arithmetical means

$$M_n(s) = \sum_{m=1}^{n} s_m/n \tag{2}$$

of $s_1, s_2, \cdots$. In fact, both (1) and (2) represent the arithmetical mean of the n values $f(1/n)s_1, f(2/n)s_2, \cdots, f(n/n)s_n$ belonging to a *fixed* non-negative, R-integrable function $f(t)$, $0 \leqq t \leqq 1$; the latter being the constant 1 in the case of (2) and the discontinuous function

$$f(t) = t^{-1} - [t^{-1}], \qquad (0 < t \leqq 1) \tag{3}$$

in the case of (1). The normalizing factor $(1 - C)^{-1}$ is introduced by the fact that, while the average of $f(t) \equiv 1$ over the interval $0 \leqq t \leqq 1$ is 1, the average of the function (3) is

$$\int_0^1 f(t)dt = \lim_{n\to\infty} \int_{1/n}^1 (t^{-1} - [t^{-1}])dt = \lim_{n\to\infty} (\log n - \sum_{m=2}^{n} m^{-1}) = 1 - C.$$

Thus, from (1),

$$D_n(1) = \sum_{m=1}^{n} (n/m - [n/m])/m \to 1 - C. \qquad (n \to \infty). \tag{4}$$

If $d(n)$ denotes the number of all divisors ($\geqq 1$) of n, then the case $s_n \equiv 1$ of the definition (1) can be written in the form

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$$-D_n(1) = \sum_{m=1}^{n} d(m)/n - \sum_{m=1}^{n} 1/m = (\sum_{m=1}^{n} d(m) - n \log n - nC + O(1))/n.$$

If this is compared with (4), it follows that the problem which, since Hardy's paper of 1915, is called *the* divisor problem of Dirichlet is equivalent to the problem of the maximum order of the remainder term of the limit relation (4), that is, to the determination of the greatest lower bound of those indices β for which $D_n(1)$ differs from the constant $1-C$ only by $O(n^{-\beta})$. In fact, Dirichlet's elementary result can be expressed by saying that this error term is $O(n^{-\frac{1}{2}})$, and about all that is known today is that the greatest lower bound in question is certainly less than 1/3 but cannot be less than 1/4 (and that it cannot be a minimum if it happens to be the optimum, 1/4).

In the present paper, a different, though from a Tauberian point of view closely related, aspect of the matrix of the transformation (1) will be investigated. If the Tauberian point of view is replaced by a certain Abelian one, then *the* divisor problem of Dirichlet ceases to be relevant for the questions considered. In fact, the latter then are connected with another divisor problem of Dirichlet, namely with the asymptotic distribution of the n fractional parts which remain when a large positive integer n is divided by any of the positive integers m not exceeding n.

The "Tauberian" and the "Abelian" connections with the respective divisor problems of Dirichlet result if the transformation of an arbitrary sequence $s_1, s_2, \cdots$ into the sequence $D_1(s), D_2(s), \cdots$ defined by (1) is thought of as defining a linear summation process (in the sense of the theory of divergent series). In fact, the "D-process" (1) represents an arithmetical counterpart of the "M-process" (2), that is, of the process of $(C,1)$-summation. However, these two processes prove to be incomparable, that is to say such that either of them can be effective when the other is ineffective.

2. The simplest fact on the linear summation process defined by (1) is that it becomes a *regular* summation process upon the insertion of the factor of proportionality assigned by (4):

(i) *If* $\lim s_n$ *exists, then* $\lim D_n(s)$ *exists and*

$$\lim D_n(s) = (1-C)^{-1} \lim s_n; \qquad C = -\Gamma'(1). \tag{5}$$

On the other hand, the existence of $\lim D_n(s)$ does not imply the existence of $\lim s_n$. More than this negation is implied by (iv) below.

If m is fixed, the n-th matrix element situated in the m-th column of the matrix of the linear substitution (1) tends to 0 as $n \to \infty$, since

$$\lim_{n\to\infty} (n/m - [n/m])/n = 1/m - 1/m = 0.$$

Furthermore, the real function (3), and therefore every element of the matrix of (1), is non-negative. Finally, (4) shows that $(1-C)^{-1}$ times the sum of all elements contained in the n-th row of the matrix of (1) tends to 1 as $n\to\infty$. Accordingly, all three conditions of Toeplitz [5] are satisfied. This proves (i).

A substantial refinement of (i) is contained in (vi) below.

The classical analogue of (i) is that convergence implies $(C,1)$-summability. The relevant analogue corresponding to Abel's theorem (that is, to the fact that convergence implies (A)-summability) naturally involves the "discrepancy between the summability of the series $\Sigma s_n/n$ in the sense of Lambert and in the sense of Abel", as follows:

(ii) *If* $\lim D_n(s)$ *exists, then both series*

$$(7_1)\quad A_r(s) = \sum_{n=1}^{\infty} s_n r^n/n; \qquad (7_2)\quad L_r(s) = \sum_{n=1}^{\infty} s_n r^n/(1-r^n)$$

converge for $r<1$ *and satisfy the limit relation*

$$(8)\qquad A_r(s) - (1-r)L_r(s) \to \lim D_n(s) \text{ as } r\to 1-0.$$

The proof of (ii) requires only an application of the classical Abelian argument (valid not only for power series but for Lambert series as well), and will therefore be omitted. Actually, generators of the type (7_1), (7_2) will not be considered in the sequel.

3. It will be shown in **5** that (i) cannot be so refined that the assumption of convergence becomes replaced by the assumption of $(C,1)$-summability, that is, the existence of $\lim s_n$ by the existence of $\lim M_n(s)$; cf. (2). All that can be said is that the summation processes defined by (1) and (2) cannot lead to contradictory evaluations:

(iii) *If both* $\lim D_n(s)$ *and* $\lim M_n(s)$ *exist, then*

$$(9)\qquad \lim D_n(s) = (1-C)^{-1}\lim M_n(s); \qquad C = -\Gamma'(1).$$

This will be verified by first calculating the elements of the infinite matrix which transforms the averages (2) into the averages (1).

To this latter end, let both sides of the definition (2) be multiplied by n. Then, if n is replaced by $n-1$, it follows by subtraction that

$$(10)\qquad s_n = nM_n(s) - (n-1)M_{n-1}(s); \qquad M_0(s) = 0.$$

But the explicit form of the linear substitution transforming the averages (2) into the averages (1) follows by substituting the case $n=m$ of (10) into the sum on the right of (1). After obvious reductions, the result of this substitution appears in the form

$$D_n(s)=\sum_{m=1}^{n-1}\alpha_{nm}M_m(s), \qquad (n>1;\ D_1(s)=0), \tag{11}$$

where the absolute constants α are given by

$$\alpha_{nm}=1/(m+1)-[n/m]m/n+[n/(m+1)]m/n \tag{12}$$

(the brackets [ ] refer to the greatest integers but $(m+1)$ is just $m+1$).

It is clear from the general theory of linear summation processes, that the consistency of the evaluations, which is the claim of (iii), is equivalent to the following *pair* of assertions (which correspond to Toeplitz's *three-fold* condition for regularity, used in **2**): The n-th matrix element situated in the m-th column of the matrix of the linear substitution (11) tends to a limit, as $n\to\infty$, for every fixed m, and the sum of all matrix elements contained in the n-th row tends to the limit $1-C$ as $n\to\infty$. But (12) shows that, if m is fixed, the n-th element of the m-th column tends to the limit $1/(m+1)$, since

$$-\lim_{n\to\infty}[n/m]m/n+\lim_{n\to\infty}[n/(m+1)]m/n=-1+1=0.$$

Hence, in order to complete the proof of (iii), it is sufficient to ascertain that the sum (11) belonging to the sequence $M_1(s)=1, M_2(s)=1,\cdots$ tends to the limit $1-C$ as $n\to\infty$. And the truth of this limit relation can be verified either from (11) or as follows:

According to (2), the sequence $M_1(s), M_2(s),\cdots$ belonging to the sequence $s_1=1, s_2=1,\cdots$ is $M_1(1)=1, M_2(1)=1,\cdots$. Hence, (11) shows that $D_n(1)$ is identical with the sum of all coefficients α_{nm} belonging to a fixed n. Consequently, the assertion is identical with the limit relation $D_n(1)\to 1-C$. But the latter is precisely (4).

4. This completes the proof of (iii). It will now be shown that neither of the assumptions of (iii) can be omitted. First, the following one of these two negations will be proved:

(iv) *The existence of* $\lim D_n(s)$ *does not imply the existence of* $\lim M_n(s)$.

In order to prove (iv), it is sufficient to show that the *inverse* of the linear transformation (11) does not satisfy the norm-condition of Lebesgue-

Toeplitz, that is, that the sum of the absolute values of the elements contained in the n-th row of the inverse matrix is not $O(1)$ as $n \to \infty$.

Actually, this method seems to fail in the present case, since the linear transformation (11) has no proper inverse, all elements of the matrix of (11) being 0 not only above, but also within, the principal diagonal. In addition, (12) shows that the only α occurring in the second row, namely α_{21}, is 0. However, if $n > 2$, then the integral part of the ratio of n either to $n-1$ or to n is 1, and so the case $m = n - 1$ of (12) shows that

$$\alpha_{n\,n-1} = 1/n, \text{ if } n > 2. \tag{13}$$

Consequently, the formal difficulty can be avoided by the following device:

If n runs through all positive integers greater than 2, and if $E_{n-1}(s)$ is an abbreviation for the difference

$$E_{n-1}(s) = D_n(s) - \alpha_{n1}M_1(s), \tag{14}$$

then (11) can be written in the form

$$E_{n-1}(s) = \sum_{m=1}^{n-1} \alpha_{nm}M_m(s), \tag{15}$$

which, since $n = 3, 4, \cdots$, represents a linear transformation of the sequence $M_2(s), M_3(s), \cdots$ into a sequence $E_2(s), E_3(s), \cdots$. The matrix elements situated above the principal diagonal of the matrix of this linear transformation are all 0, but the n-th diagonal element, being precisely the number (13), is distinct from 0 for every n. Consequently, (15) has a unique inverse of the form

$$M_{n-1}(s) = \sum_{m=2}^{n-1} \beta_{nm}E_m(s), \tag{16}$$

and the n-th diagonal element of the matrix of (16) is the reciprocal value of the n-th diagonal element of the matrix of (15).

Thus $\beta_{n\,n-1} = n$, by (13). Hence

$$\sum_{m=1}^{n-1} | \beta_{nm} | \geqq | \beta_{n\,n-1} | = n, \tag{17}$$

which is not $O(1)$ as $n \to \infty$. Consequently, if the norm-principle of Lebesgue-Toeplitz is applied to the linear transformation (16), it follows that the existence of $\lim E_n(s)$ does not imply the existence of $\lim M_n(s)$.

In order to complete the proof of (iv), it is sufficient to ascertain that the existence of $\lim E_n(s)$ is equivalent to the existence of $\lim D_n(s)$. It follows therefore from (14) that it is sufficient to verify the limit relation

$\alpha_{n1} \to 0$. But the latter is obvious from the definition (12), which indeed shows that $\lim \alpha_{n1} = 0 - 1 + 1$.

5. This concludes the proof of (iv). It will now be shown that the assertions of (iii) and (iv) can be completed as follows:

(v) *The existence of* $\lim M_n(s)$ *does not imply the existence of* $\lim D_n(s)$.

Clearly, (v) is equivalent to the statement that the norm-condition is violated by the matrix of the linear transformation (12), i. e., that

$$\sum_{m=1}^{n-1} | \alpha_{nm} | \neq O(1) \tag{18}$$

as $n \to \infty$. But (18) can be proved by an adaptation of a procedure recently applied (Wintner [6], § 10), as follows:

Let n be fixed for the present. If k is a positive integer less than n, then a positive integer m satisfies both conditions

$$[n/m] = k, \qquad [n/(m+1)] = k$$

if and only if it fulfills both inequalities

$$n/(k+1) < m \leqq n/k - 1. \tag{19}$$

Hence, if m satisfies (19) for some k, then the second term on the right of (12) cancels the third. In other words, (12) becomes $\alpha_{nm} = 1/(m+1)$ for every m for which there exists some k satisfying (19). Since the second of the inequalities (19) implies that $1/(m+1) \geqq k/n$, it follows that $\alpha_{nm} \geqq k/n$ holds for every m satisfying (19). But k in (19) was any positive integer less than n. Since it is clear that one and the same m cannot satisfy (19) for two distinct values of k (in fact, n is fixed), it follows that

$$\sum_{m=1}^{n-1} | \alpha_{nm} | \geqq \sum_{k=1}^{n-1} N_{nk} k/n, \tag{20}$$

if N_{nk} denotes the number of those positive integers m which satisfy (19).

This definition of N_{nk} implies that N_{nk} is not less than the difference between $(n/k) - 1$ and $n/(k+1)$. Since this difference can be written in the form $n/(k^2 + k) - 1$, it follows that

$$N_{nk} k/n \geqq 1/(k+1) - k/n. \tag{21}$$

On the other hand, if n is large enough, then $n-1$ exceeds $n^{\frac{1}{2}}$, and so it is clear from the inequality (20), in which every term is non-negative, that

$$\sum_{m=1}^{n-1} |\alpha_{nm}| \geqq \sum_{m=1}^{n^{\frac{1}{2}}} N_{nk}k/n \tag{22}$$

(the upper summation limit $n^{\frac{1}{2}}$ refers, of course, to the integer $[n^{\frac{1}{2}}]$).

Now let $n \to \infty$. Then, since (21) and (22) entail the inequality

$$\sum_{m=1}^{n-1} |\alpha_{nm}| \geqq \sum_{k=1}^{n^{\frac{1}{2}}} 1/(k+1) - \sum_{k=1}^{n^{\frac{1}{2}}} k/n,$$

and since

$$\sum_{k=1}^{x} k = O(x^2) \text{ as } x \to \infty,$$

it is clear that

$$\sum_{m=1}^{n-1} |\alpha_{nm}| \geqq \sum_{k=1}^{n^{\frac{1}{2}}} 1/(k+1) - O(n^{\frac{1}{2}})^2/n = \log(n^{\frac{1}{2}}) + O(1). \tag{23}$$

Since (23) implies (18), the proof of (v) is complete.

6. It is easily realized that the positive integers m for which there does not exist a positive integer k satisfying (19) for a fixed n become very scarce as $n \to \infty$. This suggests that the order of the lower estimate (23), an estimate in which the m-values violating (19) for every k (and for a fixed n) were not utilized, is substantially sharp. Actually, it is easy to prove directly that the logarithmic lower estimate (23) can be completed by the upper estimate

$$\sum_{m=1}^{n-1} |\alpha_{nm}| = O(\log n). \tag{24}$$

In fact, since both

$$\sum_{m=1}^{n-1} 1/(m+1) \text{ and } \sum_{m=1}^{n-1} [n/(m+1)]/n$$

are $O(\log n)$, it is seen from (12) that (24) is certainly true if

$$\sum_{m=1}^{n} |a_{nm}| = O(\log n) \tag{25}$$

is true, where (a_{nm}) denotes the matrix defined by

$$a_{nm} = [n/m]m/n - [n/(m+1)](m+1)/n. \tag{26}$$

The definition (26) clearly implies that

$$\sum_{m=1}^{n} a_{nm} = 1 \tag{27}$$

for every n. Hence, in order to prove (25), it is sufficient to ascertain that there exists a matrix (c_{nm}) satisfying the following three conditions:

$$\sum_{m=1}^{n} c_{nm} = O(\log n), \tag{28}$$

$a_{nm} + c_{nm} \geqq 0$ and $c_{nm} \geqq 0$. In fact, it is clear from the latter two conditions that (28) and (27) imply (25). But the existence of a matrix (c_{nm}) satisfying all three conditions is assured by the choice

$$c_{nm} = [n/(m+1)]/n. \tag{29}$$

In fact, both (28) and $c_{nm} \geqq 0$ are clear from (29) and, since

$$[n/m] \geqq [n/(m+1)],$$

it is seen from (26) and (29) that $a_{nm} + c_{nm} \geqq 0$.

This completes the proof of (24).

7. In view of (v), there arises the need for Tauberian restrictions which, when imposed on the sequence $s_1, s_2, \cdots$, necessitate the existence of $\lim D_n(s)$ whenever $\lim M_n(s)$ exists. Such a sufficient Tauberian restriction is contained in the assumption that the least upper bound of

$$\sum_{m \leqq \epsilon n} | s_m |/n$$

for $n = 1, 2, \cdots$, a least upper bound ($\leqq \infty$) representing a function of $\epsilon > 0$, should tend to 0 as $\epsilon \to 0$ (cf. Wintner [6], § 2, where the sufficiency of this assumption is proved, though not explicitly stated, as seen from the first inequality in the formula line following formula (8) on p. 4). In other words, the existence of $\lim M_n(s)$ implies the existence of $\lim D_n(s)$ whenever

$$\underset{1 \leqq n < \infty}{\text{l. u. b.}} \sum_{m=1}^{\epsilon n} | s_m |/n \to 0 \text{ as } \epsilon \to 0 \tag{30}$$

(the summation limit ϵn in (30) refers, of course, to the integer $[\epsilon n]$). This leads to the following criteria, which are Tauberian with reference to (v):

(v bis) *The existence of* $\lim D_n(s)$ *follows from the existence of* $\lim M_n(s)$ *whenever the sequence* $s_1, s_2, \cdots$ *satisfies any of the following restrictions:*

$$| s_1 | + \cdots + | s_n | = O(n); \tag{I}$$

(II) $$s_n = O_L(1);$$

(III) $$s_n = 0 \text{ unless } n = k_1, k_2, \cdots, \text{ where } k_{m+1}/k_m > \lambda > 1.$$

The sufficiency of (I), to which and (II) and (III) prove to be reducible, is due to Axer ([1]; for further references cf. [2]). Needless to say, any Tauberian theorem of type (v bis) is quite unsatisfactory without the fact (v), which does not seem to have been proved before.

It is clear that (I) is sufficient for (30).

As to (II), suppose first that $s_n \geqq 0$. Then, since $\lim M_n(s)$ is supposed to exist, it is clear from (2) that (I), hence (30), is satisfied. But the assumption (II) can be reduced to the assumption $s_n \geqq 0$, if s_n is replaced by $s_n + \text{const}$. It follows therefore from the distributive nature of both processes (1), (2), that, in order to complete the proof of the sufficiency of (II), it is sufficient to observe that, according to (4) and (2), both $\lim D_n(1) = 1 - C$ and $\lim M_n(1) = 1$ exist.

Finally, if $k_1, k_2, \cdots$ is any sequence satisfying the gap condition $k_{m+1}/k_m > \lambda$ for some fixed $\lambda > 1$, then obviously

$$\sum_{k_m \leq n} k_m = O(n).$$

On the other hand, it is clear from (2) that the existence of $\lim M_n(s)$ entails the estimate $s_n = o(n)$, hence $s_n = O(n)$, as $n \to \infty$. In particular, $s_{k_m} = O(k_m)$ holds as $m \to \infty$. Consequently, the preceding formula line implies that

$$\sum_{k_m \leq n} | s_{k_m} | = O(n).$$

But this estimate becomes identical with (I) if (III) is assumed.

This completes the proof of (v bis).

If the lower estimate (17), which belongs to (iv), is contrasted with the upper estimate (25), which belongs to (v), it is seen that Tauberian theorems which correspond to (iv) in the same way as (v bis) corresponds to (v) are likely to require an approach more elaborate than the proof of (v bis).

8. For the sake of shortness, let the linear transformation (1) of a sequence $s_1, s_2, \cdots$ into $D_1(s), D_2(s), \cdots$ be called the D-process. Thus, if (λ_{nm}) denotes the infinite square-matrix defining the D-process, then

(31) $$\lambda_{nm} = 0 \text{ for } m = n+1, n+2, \cdots$$

and

(32) $$\lambda_{nm} = (n/m - [n/m])/n \text{ for } m = 1, \cdots, n;$$

in particular, since $t - [t]$ is non-negative (and less than 1) for every t,

$$\lambda_{nm} \geqq 0, \qquad (\lambda_{nm} < 1). \tag{33}$$

It will be shown that certain elementary limit relations in the analytic theory of numbers, which go back to Dirichlet, can in the main be thought of as expressions of the fact that the D-process is "ray-like" (*gestrahlt*) in the sense of the moment theory of summation processes, as developed by Toeplitz (in an address delivered at Leipzig in 1922; cf. R. Schmidt [4], p. 94) and, at his suggestion, further analyzed by R. Schmidt [4].

(vi) *The matrix of the D-process is a "ray-like" (gestrahlt) matrix. Furthermore, it is a moment matrix and has the unique weight function*

$$\phi(x) = \begin{cases} \phi(1) = 1 - C \text{ if } 1 < x < \infty; \\ \int_0^x (t^{-1} - [t^{-1}])dt \text{ if } 0 \leqq x \leqq 1. \end{cases} \tag{34}$$

In particular,

$$\phi(0) = \phi(+0). \tag{35}$$

Finally, the moment function, $\mu(\sigma)$, of the D-process is expressible in terms of Riemann's zeta-function, as follows:

$$\mu(\sigma) = \begin{cases} (\sigma - 1)^{-1} - \zeta(\sigma)/\sigma \text{ if } 0 < \sigma < \infty; \\ \mu(+0) = 1 - C \text{ if } \sigma = 0. \end{cases} \tag{36}$$

According to the general theory (cf. R. Schmidt [4], Theorem II), the assertions of (vi) imply the following corollary:

COROLLARY 1. *If a sequence $s_1, s_2, \cdots$ is "ray-like" (gestrahlt), then the same is true of the sequence $D_1(s), D_2(s), \cdots$ of its transforms* (1) *and the latter satisfy the relation $D_n(s)/s_n \to \mu(\sigma)$ as $n \to \infty$, where σ denotes the convergence exponent of $s_1, s_2, \cdots$ and $\mu(\sigma)$ is the corresponding positive constant* (36).

(It is due to (35) that Corollary 1 need not exclude the case $\sigma = 0$).

COROLLARY 2. *If a sequence $s_1, s_2, \cdots$ is "very slowly oscillating," then the same is true of the sequence $D_1(s), D_2(s), \cdots$ of its transforms* (1).

This follows from (vi) and (31), if use is made of one of R. Schmidt's principal results, namely of his Theorem V. If the latter is replaced by his Theorem VI, then (vi) supplies the following fact:

COROLLARY 3. *If a sequence* $s_1, s_2, \cdots$ *is "slowly oscillating," then the same is true of the sequence* $D_1(s), D_2(s), \cdots$ *of its transforms* (1).

In order to verify (vi), let $f_x(t)$, where $x \geqq 0$ is fixed and $0 \leqq t < \infty$, denote the function defined by

$$(37) \qquad f_x(t) = \begin{cases} t^{-1} - [t^{-1}] \text{ if } 0 < t \leqq \operatorname{Min}(1, x); \\ \quad 0 \text{ if } \operatorname{Min}(1, x) < t < \infty. \end{cases} \qquad f_x(0) = 0;$$

Then it is clear from (31) and (32) that

$$(38) \qquad \sum_{m \leqq xn} \lambda_{nm} = \sum_{m=1}^{n} f_y(m/n)/n,$$

where

$$(39) \qquad y = \operatorname{Min}(1, x).$$

Since (37) and (39) imply that $f_y(t)$ is an R-integrable function on the interval $0 \leqq t \leqq 1$, it is seen from (38) that

$$(40) \qquad \sum_{m \leqq nx} \lambda_{nm} \to \int_0^1 f_y(t)\, dt \text{ as } n \to \infty.$$

But it is clear from (39) and (37) that the integral on the right of (40) is identical with the integral on the right of (34) or with the constant $1 - C$ on the right of (34) according as $x \leqq 1$ or $x \geqq 1$.

In view of the definition of the notions occurring in (vi), this proves all the assertions of (vi), except the evaluation (36). The latter can be verified in various ways, for instance as follows: A partial Stieltjes integration shows that

$$\sum_{n=1}^{\infty} a_n n^{-\sigma} = \sigma \int_1^{\infty} \Big(\sum_{n \leqq x} a_n\Big) x^{-\sigma-1}\, dx$$

holds for $\sigma > 1$, if the Dirichlet series on the left converges for $\sigma > 1$. If this is applied to the case $a_n = 1$ of Riemann's zeta-function, it is seen from the identity

$$(\sigma - 1)^{-1} = \int_1^{\infty} x^{-\sigma}\, dx, \qquad (\sigma > 1),$$

that, since $\sum_{n \leqq x} 1 = [x]$,

$$(\sigma - 1)^{-1} - \zeta(\sigma)/\sigma = \int_1^{\infty} (x - [x]) x^{-\sigma-1}\, dx$$

holds for $\sigma > 1$ and so, for reasons of analyticity, for $\sigma > 0$ (de la Vallée-

Poussin). Hence, if the integration variable x is replaced by x^{-1}, it follows for $\sigma > 0$ that

$$(\sigma - 1)^{-1} - \zeta(\sigma)/\sigma = \int_0^1 (x^{-1} - [x^{-1}])x^\sigma \, dx \equiv \int_0^\infty x^\sigma \, d\phi(x),$$

by (34). This proves the first line of (36). The second line of (36) follows either from (4) or, for reasons of continuity, from the fact that the constant term of the power series of the entire function $\zeta(s) - (s-1)^{-1}$ at the point $s = 1$ is the Euler-Mascheroni constant.

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REFERENCES

[1] A. Axer, "Beitrag zur Kenntnis der zahlentheoretischen Funktionen $\mu(n)$ und $\lambda(n)$," *Prace Mat.-Fiz.*, vol. 21 (1910), pp. 65-96.
[2] H. Bohr und H. Cramér, *Enc. der math. Wiss.*, vol. II C (1922), pp. 814-815.
[3] G. P. Lejeune Dirichlet, *Werke*, vol. 2, pp. 51-66 and pp. 99-104
[4] R. Schmidt, "Ueber divergente Folgen und lineare Mittelbildungen," *Math. Ztschr.*, vol. 22 (1925), pp. 89-152.
[5] O. Toeplitz, "Ueber allgemeine lineare Mittelbildungen," *Prace Mat.-Fiz.*, vol. 22 (1910), pp. 113-119.
[6] A. Wintner, *Eratosthenian Averages*, Baltimore, 1943.

HOW FAR CAN ONE GET WITH A LINEAR FIELD THEORY OF GRAVITATION IN FLAT SPACE-TIME? *

By Hermann Weyl.

Introduction and Summary. G. D. Birkhoff's attempt to establish a linear field theory of gravitation within the frame of special relativity [1] makes it desirable to probe the potentialities and limitations of such a theory in more general terms. In thus continuing a discussion begun at another place [2] I find that the differential operators at one's disposal form a 5 dimensional linear manifold. But the requirement that the field equations imply the law of conservation of energy and momentum in the simple form $\partial T_i^k/\partial x_k = 0$ limit these ∞^5 possibilities to ∞^2, which, however, reduce easily to two cases, a regular one (L) and a singular one (L'). The regular case (L) is nothing but Einstein's theory of weak fields. Resembling very closely Maxwell's theory of the electromagnetic field, it satisfies a principle of gauge invariance involving 4 arbitrary functions, and although its gravitational field exerts no force on matter, it is well suited to illustrate the role of energy and momentum, charge and mass in the interplay between matter and field. It might also help, though this is much more problematic, in pointing the way to a more satisfactory unification of gravitation and electricity than we at present possess. Birkhoff follows the opposite way: by avoiding rather than adopting the ∞^2 special operators mentioned above, his "dualistic" theory (B) destroys the bond between mechanical and field equations, which is such a decisive feature in Einstein's theory.

1. Maxwell's theory of the electromagnetic field and the monistic linear theory of gravitation (*L*). Gauge invariance. Within the frame of special relativity and its metric ground form

$$ds^2 = \delta_{ik} dx_i dx_k = dx_0^2 - (dx_1^2 + dx_2^2 + dx_3^2)$$

an electromagnetic field is described by a skew tensor

$$f_{ik} = \partial\phi_k/\partial x_i - \partial\phi_i/\partial x_k$$

derived from a vector potential ϕ_i and satisfies Maxwell's equations

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* Received August 9, 1944.

[1] *Proceedings of the National Academy of Sciences*, vol. 29 (1943), p. 231.

[2] *Proceedings of the National Academy of Sciences*, vol. 30 (1944), p. 205.

$$(1) \qquad \partial f^{ki}/\partial x_k = s^i \text{ or } D_i\phi = \Box\phi_i - \partial\phi'/\partial x_i = s_i$$

where s^i is the density-flow of electric charge and

$$\phi' = \partial\phi^i/\partial x_i, \qquad \Box\phi = \delta^{pq}(\partial^2\phi/\partial x_p\partial x_q).$$

The equations do not change if one substitutes

$$(2) \qquad \phi^*_i = \phi_i - \partial\lambda/\partial x_i \quad \text{for} \quad \phi_i,$$

λ being an arbitrary function of the coördinates ("*gauge invariance*"), and they imply the differential conservation law of electric charge:

$$(3) \qquad \partial s^i/\partial x_i = 0.$$

As is easily verified, there are only two ways in which one may form a vector field by linear combination of the second derivatives of a given vector field ϕ_i, namely,

$$\Box\phi_i \text{ and } \partial\phi'/\partial x_i \qquad (\phi' = \partial\phi^p/\partial x_p).$$

The only linear combination $D_i\phi$ of these two vector fields which satisfies the identity $(\partial/\partial x_i)(D^i\phi) = 0$ is the one occurring in (1),

$$D_i\phi = \Box\phi_i - \partial\phi'/\partial x_i.$$

Herein lies a sort of mathematical justification for Maxwell's equations.

Taking from Einstein's theory of gravitation the hint that gravitation is represented by a symmetric tensor potential h_{ik}, but trying to emulate the linear character of Maxwell's theory of the electromagnetic field, one could ask oneself what symmetric tensors $\tilde{D}_{ik}h$ can be constructed by linear combination from the second derivatives of h_{ik}. The answer is that there are 5 such expressions, namely

$$(4) \qquad \Box h_{ik}, \quad \partial h'_i/\partial x_k + \partial h'_k/\partial x_i, \quad h''\delta_{ik}, \quad \partial^2 h/\partial x_i\partial x_k, \quad \Box h\cdot\delta_{ik}$$

where

$$h = h_p^p, \qquad h'_i = \partial h_i^p/\partial x_p, \qquad h'' = \partial^2 h^{pq}/\partial x_p\partial x_q.$$

With any linear combination $\tilde{D}_{ik}h$ of these 5 expressions one could set up the field equations of gravitation

$$(\tilde{5}) \qquad \tilde{D}_{ik}h = T_{ik}$$

the right member of which is the energy-momentum tensor T_{ik}. In analogy

to the situation encountered in Maxwell's theory one may ask further for which linear combinations $\tilde{D}_{ik}$ the identity

$$(\partial/\partial x_k)(\tilde{D}_i{}^k h) = 0$$

will hold, and one finds that this is the case if, and only if, $\tilde{D}_{ik}h$ is of the form

$$(6) \qquad \alpha\{\Box h_{ik} - (\partial h'_i/\partial x_k + \partial h'_k/\partial x_i) + h''\delta_{ik}\} + \beta\{\partial^2 h/\partial x_i \partial x_k - \Box h \cdot \delta_{ik}\},$$

α and β being arbitrary constants. In this case the field equations ($\tilde{5}$) entail the differential conservation law of energy and momentum

$$(7) \qquad \partial T_i{}^k/\partial x_k = 0.$$

With two constants a, b ($a \neq 0$, $a \neq 4b$) we can make the substitution

$$h_{ik} \to a \cdot h_{ik} - b \cdot h\delta_{ik}$$

and thereby reduce α, β to the values 1, 1, provided $\alpha \neq 0$, $\alpha \neq 2\beta$. Hence, disregarding these singular values, we may assume as our field equations

$$(5) \qquad \begin{aligned} D_{ik}h \equiv \{\Box h_{ik} - (\partial h'_i/\partial x_k + \partial h'_k/\partial x_i) + h''\delta_{ik}\} \\ + \{\partial^2 h/\partial x_i \partial x_k - \Box h \cdot \delta_{ik}\} = T_{ik}. \end{aligned}$$

$D_{ik}h$ remains unchanged if h_{ik} is replaced by

$$(8) \qquad h^*_{ik} = h_{ik} + (\partial\xi_i/\partial x_k + \partial\xi_k/\partial x_i)$$

where ξ_i is an arbitrary vector field. Hence we have the same type of correlation between gauge invariance and conservation law for the gravitational field as for the electromagnetic field, and it is reasonable to consider as physically equivalent any two tensor fields h, h^* which are related by (8).

The linear theory of gravitation (L) in a flat world at which one thus arrives with a certain mathematical necessity is nothing else but Einstein's theory for weak fields. Indeed, on replacing Einstein's g_{ik} by $\delta_{ik} + 2\kappa \cdot h_{ik}$ and then neglecting higher powers of the gravitational constant κ, one obtains (5), and the property of gauge invariance (8) reflects the invariance of Einstein's equations with respect to arbitrary coördinate transformations.[3]

By proper normalization of the arbitrary function λ in (2) one may impose the condition $\phi' = 0$ upon the ϕ_i, thus giving Maxwell's equations a form often used by H. A. Lorentz:

$$(9) \qquad \Box\phi_i = s_i, \qquad \partial\phi^i/\partial x_i = 0.$$

[3] Cf. A. Einstein, *Sitzungsber. Preuss. Ak. Wiss.* (1916), p. 688 (and 1918, p. 154).

In the same manner one can choose the ξ_i in (8) so that $\gamma_{ik} = h_{ik} - \frac{1}{2}h \cdot \delta_{ik}$ satisfies the equations

$$(10) \qquad \partial \gamma_i{}^k / \partial x_k = 0 \text{ and}$$

$$(11) \qquad \Box \gamma_{ik} = T_{ik}.$$

In one important respect gauge invariance works differently for electromagnetic and gravitational fields: If one splits the tensor of derivatives $\phi_{k,i} = \partial \phi_k / \partial x_i$ into a skew and a symmetric part,

$$\phi_{k,i} = \tfrac{1}{2}(\phi_{k,i} - \phi_{i,k}) + \tfrac{1}{2}(\phi_{k,i} + \phi_{i,k}),$$

the first part is not affected by a gauge transformation whereas the second can locally be transformed into zero. In the gravitational case *all* derivatives $\partial h_{ik}/\partial x_p$ can locally be transformed into zero. Hence we may construct, according to Faraday and Maxwell, an energy-momentum tensor L_{ik} of the electromagnetic field,

$$(12) \qquad L_i{}^k = f_{ip} f^{pk} - \tfrac{1}{2}\delta_i{}^k (ff), \qquad (ff) = \tfrac{1}{2} f_{pq} f^{qp},$$

depending quadratically on the gauge invariant field components

$$f_{ik} = \phi_{k,i} - \phi_{i,k},$$

but no tensor G_{ik} depending quadratically on the derivatives $\partial h_{ik}/\partial x_p$ exists, if gauge invariance is required, other than the trivial $G_{ik} \equiv 0$.

2. Particles as centers of force, and the charge vector and energy-momentum tensor of a continuous cloud of substance. Conceiving a resting particle as a center of force, let us determine the *static centrally symmetric solutions* of our homogeneous field equations (1) and (5) ($s^i = 0$, $T_{ik} = 0$). One easily verifies that *in the sense of equivalence* the most general such solution is given by the equations

$$(13) \qquad \phi_0 = e/4\pi r, \qquad \phi_i = 0 \quad \text{for } i \neq 0;$$

$$(14) \qquad \gamma_{00} = m/4\pi r, \qquad \gamma_{ik} = 0 \text{ for } (i,k) \neq (0,0),$$

r being the distance from the center. As was to be hoped, it involves but two constants, *charge* e and *mass* m. The center itself appears as a singularity in the field. Indeed ϕ_0 and the factor ϕ in $\phi_\alpha = \phi x_\alpha$ $[\alpha = 1, 2, 3]$ must be functions of r alone, and the relations

$$\Delta\phi_0 = 0, \qquad \partial\phi_a/\partial x_a = 0 \qquad [\alpha = 1, 2, 3]$$

implied in (9) then yield

$$\phi_0 = a/r, \qquad \phi = b/r^3, \qquad \phi_a = -(\partial/\partial x_a)(b/r).$$

Substitution of $\phi_a - \partial\lambda/\partial x_a$ for ϕ_a with $\lambda = -b/r$ changes ϕ_a into zero. In the same manner (14) is obtained from the equations (10 & 11).

A continuous cloud of "charged dust" can be characterized by its velocity field u^i ($u_i u^i = 1$) and the rest densities μ, ρ of mass and charge. It is well known that its equations of motion and the differential conservation laws of mass and charge result if one sets $s^i = \rho u^i$ in Maxwell's equations and lets T_i^k in (7) consist of the Faraday-Maxwell field part (12) and the kinetic part $\mu u_i u^k$:

$$\partial(\rho u^i)/\partial x_i = 0, \qquad \partial(\mu u^i)/\partial x_i = 0;$$
$$\mu du_i/ds = \rho \cdot f_{ip} u^p.$$

Since the motion of the individual dust particle is determined by $dx_i/ds = u^i$ we have written d/ds for $u^k \partial/\partial x_k$. In this manner Faraday explained by his electromagnetic tensions (flow of momentum) the fact that the *active* charge which generates an electric field is at the same time the *passive* charge on which a given field acts. At its present stage our theory (L) accounts for the force which an electromagnetic field exerts upon matter, but the gravitational field remains a powerless shadow. From the standpoint of Einstein's theory this is as it should be, because the gravitational force arises only when one continues the approximation beyond the linear stage. We pointed out above that no remedy for this defect may be found in a gauge invariant gravitational energy-momentum tensor. However, the theory (L) explains why active gravity, represented by the scalar factor μ in the kinetic term $\mu u_i u_k$ as it appears in the right member T_{ik} of the gravitational equations (5), is at the same time inertial mass: this is simply another expression of the fact that the mechanical equations (7) are a consequence of those field equations.

We have seen that even in empty space the field part of energy and momentum must not be ignored, and thus a particle should be described by the static centrally symmetric solution of the equations

$$(15) \qquad D_i\phi = 0, \qquad D_{ik}h - L_{ik} = 0$$

(of which the second set is no longer strictly linear!). Again we find, after proper gauge normalization,

$$\phi_0 = e/4\pi r, \qquad \phi_1 = \phi_2 = \phi_3 = 0, \tag{13}$$

and then

$$\left\{ \begin{array}{ll} \gamma_{00} = m/4\pi r - \frac{1}{4}(e/4\pi r)^2, & \gamma_{0\alpha} = 0, \\ \gamma_{\alpha\beta} = -(e/4\pi r)^2 \cdot (x_\alpha x_\beta/4r^2) & [\alpha, \beta = 1, 2, 3]. \end{array} \right. \tag{14_e}$$

As before, two characteristic constants e and m appear. *At distances much larger than the "radius" $e^2/4\pi m$ of the particle the gravitational influence of charge becomes negligible compared with that of mass.*

3. The singular case. In normalizing the operator (6) by $\alpha = \beta = 1$ we had to exclude the cases $\alpha = 0$, $\beta = 1$ and $\alpha = 1$, $\beta = 1/2$. The first is clearly without interest because it deals with a field described by a scalar h rather than a tensor h_{ik}. But the differential operator (6), D'_{ik}, corresponding to the values $\alpha = 1$, $\beta = 1/2$ and the attendant field equations

$$D'_{ik}h = T_{ik} \tag{5'}$$

deserve a moment's attention. $D'_{ik}h$ remains unchanged if h_{ik} is replaced by

$$h^*_{ik} = h_{ik} + \eta\delta_{ik} + (\partial\xi_i/\partial x_k + \partial\xi_k/\partial x_i)$$

where the 5 functions η, ξ_i are subject to the one restriction $\partial\xi^i/\partial x_i = 0$. By proper gauge normalization one may reduce the field equations (5′) to the form

$$\partial h_i{}^k/\partial x_k = 0, \tag{10'}$$

$$\Box h_{ik} + \tfrac{1}{2}(\partial^2 h/\partial x_i \partial x_k - \Box h \cdot \delta_{ik}) = T_{ik}. \tag{11'}$$

The static centrally symmetric solution of the homogeneous equations ($T_{ik} = 0$) is the following counterpart to (14):

$$h_{00} = 0, \quad h_{0\alpha} = 0, \quad h_{\alpha\beta} = (m'/4\pi r)(\delta_{\alpha\beta} - x_\alpha x_\beta/r^2) \quad [\alpha, \beta = 1, 2, 3]$$

The same electric part as in (14_e) may be superimposed. It seems remarkable that besides (L) this possibility (L') exists.

4. Derivation of the mechanical laws wihtout hypotheses about the inner structure of particles. In principle the idea of substance had already been overcome by Newton's dynamical interpretation of Nature. His particles are centers of force, the inertial mass is a dynamic coefficient and not, as the scholastic definition pretends, quantity of substance. Boscovich, Ampère and others took the extreme view that the centers of force are points without extension. Modern atomistic physics has raised the discrete structure of matter

above all doubt. Although it does not forbid us to picture the elementary particles as something of continuous extension, one must admit that, so far, speculations about their "interior" have never borne fruit. Indeed we can explain the laws of reaction of particles with the continuous field without committing ourselves to any hypotheses concerning their inner structure, simply *by describing a particle through the surrounding "local" field.* I proceed to illustrate this fundamental point first by Maxwell's equations and then by our linear theory (L).

A particle describes a narrow channel in the 4 dimensional world. The only assumption concerning the electromagnetic potential ϕ_i we make is that outside this channel Maxwell's homogeneous equations

$$\partial f^{ki}/\partial x_k = 0 \tag{16}$$

are satisfied. By arbitrary continuous extension we fill the channel with a *fictitious field* ϕ_i and then *define* s^i by (1). The relation (3) is a consequence of this definition, and (16) asserts that s^i vanishes outside the channel. Let S_t denote the plane $x_0 = \text{const.} = t$, S^*_t the portion of S_t inside the channel, Ω the surface of the channel and Ω_t the intersection of Ω with S_t (or the boundary of S^*_t). The surface Ω_t surrounds the particle in the 3-space S_t. Integrating (3) over S_t we find

$$de/dt = 0 \quad \text{for} \quad e = \iiint_{S^*_t} s^0 dx_1 dx_2 dx_3;$$

hence e does not vary in time. More generally, it can be stated that the vector field s^i sends the same flow e through any 3 dimensional surface crossing the channel. Application of this fact to two different cross sections S_t confirms the above result; application to two cross sections $x_0 = \text{const.}$ and $x^*_0 = \text{const.}$ corresponding to two different admissible coördinate systems x and x^* (which are linked by a Lorentz transformation) proves e to be an *invariant*. Finally we must show that it is independent of the fictitious "filling." But according to the definition of s^0,

$$e = \iiint_{S^*_t} (\partial f^{01}/\partial x_1 + \partial f^{02}/\partial x_2 + \partial f^{03}/\partial x_3)\, dx_1 dx_2 dx_3$$

is the flow of the electric field (f^{01}, f^{02}, f^{03}) through Ω_t and hence is completely determined by the *real* field on Ω. For this introduction of the charge e it does not matter whether the particle is an actual singularity of the field or covers a (small) region where the known laws in empty space are suspended (and unknown laws take their place). If the field surrounding the particle is

described by (13) then the flow e of the electric field through Ω_t is the constant designated by the same letter in (13). Approximately one can ascribe a world direction u^i to the channel, and it is clear that, if numerous particles of nearly the same velocity u^i, each with its charge e, are encountered in a macroscopic "volume element" of space, their effect can macroscopically be accounted for by a convective current ρu^i.

Faute de mieux, H. A. Lorentz and H. Poincaré used this expression also for the infinitesimal volume elements of an electron, and the question arose by what cohesive forces the charges of the several parts of an electron are held together against their electrostatic repulsion. Compared with this primitive viewpoint (which was elaborated in considerable detail by M. Abraham) G. Mie's field theory of paricles,[4] which expressed the current s^i in terms of the same fundamental quantities, namely ϕ_i, as the field itself, signified an enormous progress. But also this theory, in spite of some highly attractive features, the great hopes it once raised and its development by men like D. Hilbert, M. Born and others, has remained in the limbo of speculative physics. The sober non-committal attitude here described was the third stage in the history of our problem. [A fourth has been opened by quantum physics: Following in Schrödinger's footsteps, Dirac expressed s^i in terms of the 4 spinor components of the electronic field ψ. This is a simple extension of the scheme of field physics, which in itself is as natural as the appearance of the Maxwellian L_{ik} in the gravitational field equations (15). However an entirely new feature, statistical interpretation based on quantization of the field laws, "creates" in quantum physics the discrete particles. The singularities to which this process of quantization gives rise constitute a difficulty at least as serious in quantum as in classical physics.]

Let us return to the classical standpoint and proceed from electricity to gravitation. After bridging the channel by a fictitious field h_{ik} we integrate the identities

$$(\partial/\partial x_k)(D_i{}^k h) = 0$$

over a cross section $S^*{}_t$ of the channel, thus obtaining the mechanical equations

$$dJ_i/dt = P_i \tag{16}$$

in which

$$J_i = \int\!\!\int\!\!\int_{S^*_t} D_i{}^0 h \cdot dx_1 dx_2 dx_3$$

and $-P_i$ is the flow of the vector field $(D_i{}^1 h,\ D_i{}^2 h,\ D_i{}^3 h)$ on S_t through Ω_t.

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[4] *Ann. d. Phys.*, vols. 37, 39, 40 (1912/13).

By its definition P_i does not depend on the fictitious filling, and from this fact and (16) it follows that the same is true for J_i. Indeed define $J_i^{(1)}$ by a filling 1, $J_i^{(2)}$ by a filling 2, consider two distinct cross sections S_1, S_2, $t = t_1$ and $t = t_2$, and construct a filling 3 that coincides with 1 in the neighborhood of S_1, with 2 in the neighborhood of S_2. Applying (16) to these three fillings and recalling that P_i remains unaffected one finds

$$J_i^{(1)}(t_2) - J_i^{(1)}(t_1) = \int_{t_1}^{t_2} P_i dt, \qquad J_i^{(2)}(t_2) - J_i^{(2)}(t_1) = \int_{t_1}^{t_2} P_i dt,$$

$$J_i^{(3)}(t_2) - J_i^{(3)}(t_1) = J_i^{(2)}(t_2) - J_i^{(1)}(t_1) = \int_{t_1}^{t_2} P_i dt;$$

hence

$$J_i^{(1)}(t) = J_i^{(2)}(t) \quad \text{for} \quad t = t_1 \text{ and } t_2.$$

When dealing with an *isolated* system we can assume that $D_{ik}h$ vanishes outside the channel; then $P_i = 0$. Let us choose an arbitrary constant contravariant vector l^i and form the vector field $q^k = l^i \cdot D_i^k h$, which satisfies the equation $\partial q^k / \partial x_k = 0$ and under our assumption vanishes outside the channel. The argument previously applied to s^k proves that

$$\int\int\int_{S^*_t} q^0 dx_1 dx_2 dx_3 = l^i J_i$$

is constant in time and an invariant. Hence *J_i are the components of a covariant vector.* In this way we introduce the energy-momentum vector J of an isolated particle and obtain the conservation law

$$J_i = \text{const.} \tag{17}$$

For the *static* field (14) one may compute J_i by means of a *static* filling. Then $J_1 = J_2 = J_3 = 0$ and J_0 is the integral of

$$D_0^0 h = -\Delta\gamma_{00} + \partial^2\gamma^{\alpha\beta}/\partial x_\alpha \partial x_\beta \qquad [\alpha, \beta = 1, 2, 3]$$

over a sphere S^*_0 around the center, hence the flow through its surface Ω_0 of the spatial vector

$$-\{\partial\gamma_{00}/\partial x_\alpha + \partial\gamma_\alpha{}^\beta/\partial x_\beta\}.$$

But this flow may be computed from the *real* field and thus turns out to be radial and of strength $m/4\pi \cdot 1/r^2$; consequently $J_0 = m$.

Since J_i is a covariant vector, our result $J_0 = m$, $J_1 = J_2 = J_3 = 0$ carries over from a resting isolated particle to one moving in the direction u^i:

(18) $$J_i = mu_i.$$

For a particle interacting with other particles we can not assume that $D_{ik}h$ vanishes outside the channel, and the conservation law (17) must be replaced by the mechanical equations (16). We might call P external force and J energy-momentum; both, as we have seen, are independent of the filling, but there is no reason why J should be a vector. We get beyond this general scheme by an approximate evaluation of P and J, based on the field equations (15) which hold outside Ω and the character of the local field surrounding the particle. Computation of J_0 for the static centrally symmetric field (14_e) by the same method as for the special case $e = 0$ yields

$$J_0 = m - \tfrac{1}{2} \cdot (e^2/4\pi a),$$

provided Ω_0 is the sphere of radius a. Notice that $J_0(a)$ tends to $-\infty$ and not to zero with $a \to 0$. The energy between two spheres of different radii a has the correct value of the electric field energy $(e^2/8\pi)[1/a]$; *nevertheless the total energy* ($a \to \infty$) *is not infinite but* m.

The electric field will be a superposition of the local fields generated by the several particles. In terms of a suitable system of coördinates in which the particle under consideration momentarily (for $t = 0$) rests we shall, therefore, have a field $F_{ik} + f_{ik}$ on $\Omega_0 = \Omega_{t=0}$ where

$$(f_{01}, f_{02}, f_{03}) = (e/4\pi r^3)(x_1, x_2, x_3), \qquad f_{12} = f_{23} = f_{31} = 0,$$

while F_{ik} is practically constant, i.e. varies on Ω_0 essentially less than f_{ik} (though it may well be stronger than f_{ik}). A familiar calculation then gives for the flow of

$$-(D_i{}^1h, D_i{}^2h, D_i{}^3h) = -(L_i{}^1, L_i{}^2, L_i{}^3)$$

the value $P_i = eF_{i0}$.

Were f_{ik} the total electric field we could assume that the (local) gravitational field surrounding the particle, for $t = 0$ and outside Ω_0, is given by (14_e), and we should obtain

(19) $$J_0 = m, \qquad J_1 = J_2 = J_3 = 0,$$

provided *the radius a of the sphere Ω_0 is large in comparison with the radius $e^2/4\pi m$ of the particle.* We fix Ω_0 in this manner: it is at this point that the necessity for keeping away from the particle arises. The equations (14) will still hold with sufficient accuracy on and outside Ω_0 if not only e^2/a^4 but also *the energy of the "outer" field $\Sigma F_{ik}{}^2$ on Ω_0 is small compared to m/a^3.*

Cut the channel by two cross sections $x_0 = \text{const.}$, $x^*_0 = \text{const.}$, belonging to two different coördinate systems x, x^* and going through a common point inside the channel. Let l again be an arbitrary constant contravariant vector with the components l^i in the one, l^{*i} in the other coördinate system. The difference of the respective integrals l^iJ_i, $l^{*i}J^*_i$ is the flow of

$$(l^i \cdot D_i^1 h,\ l^i \cdot D_i^2 h,\ l^i \cdot D_i^3 h) = (l^iL_i^1,\ l^iL_i^2,\ l^iL_i^3)$$

through the part of the channel surface Ω between these two cross cuts, and hence, under the above assumptions, of a lower order of magnitude than m. *With this approximation J_i is a covariant vector,* and thus the formula (18) becomes applicable not only for the cross section $t = 0$ where the particle rests momentarily, but for any cross section $x_0 = t = \text{const.}$

Of course, (16) has to be interpreted in integral fashion,

$$[J_i] = J_i(t_1) - J_i(0) = \int_0^{t_1} P_i dt,$$

and here we may set, with sufficient approximation, $J_i(t) = m(t)u_i(t)$. The equation itself shows that an appreciable change of J_i, one that is comparable with m, can be expected only after a lapse of time t_1 of order $m/e \mid F \mid$, which is large in comparison with the radius a of Ω_0: Our assumptions imply that J_i or m and u^i change but slowly (*quasi-stationary motion*).

But with these precautions in mind, the differential equation

$$(d/dt)(mu_i) = eF_{i0} \tag{20}$$

may now be claimed as holding for $t = 0$. The component $i = 0$ gives $dm/dt = 0$; hence *the mass m stays constant.* By a known simple technique (20) is changed into its invariant form

$$mdu_i/ds = e \cdot F_{ip}u^p$$

which will hold along the entire channel. The deduction indicates clearly the hypotheses to which the approximate validity of this Lorentz equation of motion of a particle is bound.[5] We now understand why quantities of the type $s^i = \rho u^i$, $T_i^k = \mu u_i u^k$ can account in a rough manner for the interaction between field and a cloud of charged dust in which near particles have nearly the same velocity.

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[5] I have repeated here for the linear theory an argument which I first developed within the frame of general relativity in the 4th and in more detail in the 5th edition of my book "Raum Zeit Materie"; see the latter edition, Berlin 1923, pp. 277-286. The purely gravitational case was treated with the greatest care in a more recent paper by A. Einstein, L. Infeld and B. Hoffmann, *Annals of Mathematics*, vol. 39 (1938), pp. 65-100.

5. Vague suggestions about a future unification of gravitation and electromagnetism. In spite of such achievements nobody will believe in the sufficiency of the linear theory (L). For, as we said above, its gravitational field is a shadow without power. The fundamental fact that *passive gravity and inertial mass* always coincide appears to me convincing proof that *general relativity* is the only remedy for this shortcoming. But thereby the gravitational constant κ enters the picture, and one knows that the ratio of the electric and gravitational radii of an electron, $(e^2/m) : \kappa m = e^2/\kappa m^2$, is a pure number of the order of magnitude 10^{40}. This circumstance and Mach's old idea that the plane of the Foucault pendulum is carried around by the stars in their daily revolution, point to a construction in which the gravitational force is bound to the totality of masses in the universe. Our present theory, Maxwell + Einstein, with its inorganic juxtaposition of electromagnetism and gravitation, cannot be the last word. Such juxtaposition may be tolerable for the linear approximation (L) but not in the final generally relativistic theory. Transition from (L) with its flat world to general relativity should raise both, not only the gravitational, but also the electromagnetic part, above the linear level and, as it changes the gauge transformations of the former into non-linear transformations of coördinates, something similar ought to happen to the gauge transformations of the ϕ_i.

After adding Dirac's 4 spin components of the electronic field ψ to the fundamental field quantities ϕ_i, h_{ik} the electric gauge invariance[6] states that the field equations do not change under the substitution of

$$e^{i\lambda}\cdot\psi, \qquad \phi_k - \frac{h}{e}\frac{\partial\lambda}{\partial x_k} \quad \text{for} \quad \psi, \phi_k$$

(h = Planck's quantum of action): the process of "covariant derivation" of ψ is defined by $\partial/\partial x_k + (ie/h)\phi_k$. Thus the electromagnetic field ϕ_i appears as a sort of appendage of the ψ-field. It is natural to expect the h_{ik} to be appended in a similar manner to quantities associated with other elementary particles. Thus incompleteness of our present theory on the linear level, a *premature* transition to general relativity, might have their share in blocking the view towards a satisfactory unification. For these reasons a linear theory of gravitation like (L), though necessarily preliminary in character, may still deserve the physicist's attention.

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[6] This principle of gauge invariance is analogous to one by which the author in 1918 made the first attempt at a unification of electromagnetism and gravitation. He has long since realized that it does not connect electricity and gravitation (ϕ_i and g_{ik}), as he then believed, but the electric with the electronic field (ϕ_i with ψ). In this form, in which the exponent of the gauge factor $e^{i\lambda}$ is pure imaginary and not real, it expresses well established atomistic facts, and the connecting coefficient, h/e, is a known atomistic and not an unknown cosmologic constant.

6. A free paraphrase of Birkhoff's recent linear theory of gravitation (*B*). The linear theory (B), however, is essentially different from (L). It seems to me characteristic for Birkhoff's conception that he uses the kinetic quantities $s^i = \rho u^i$, $T_i^k = \mu u_i u^k$ not only for a macroscopic description of matter, but a late follower of Lord Kelvin, even for the construction of fluid models of atoms, and that he preserves the duality of field and matter also in the form of mechanical equations which do not follow from the field equations. In contrast to this "dualistic" scheme Einstein's theory and its linear approximation (L) are "monistic."

Since Birkhoff wishes to avoid the fact that mechanical equations such as (7) follow from the field equations, he must choose for the left side $\tilde{D}_{ik}h$ of his linear field equations $(\tilde{5})$ any combination of the 5 tensors (4) which is *not* of the special form (6). He picks, somewhat arbitrarily, $\Box h_{ik}$ or rather $\Box h_{ik} - \frac{1}{2}\Box h \cdot \delta_{ik}$; but it seems wiser not to commit oneself too early. He is then at liberty to add to the left member of (7) a term representing the action of the gravitational field on matter. Assuming that force to be quadratic in u^i, as in Einstein's theory, he writes

$$(21) \qquad (\partial/\partial x_k)(\mu u_i u^k + L_i^k) + \Gamma_{i,pq} u^p u^q = 0$$

and finds

$$(22) \qquad \Gamma_{i,pq} = (\sigma/2)(\partial h_{ip}/\partial x_q + \partial h_{iq}/\partial x_p - 2\partial h_{pq}/\partial x_i)$$

as the mathematically simplest expression by which the differential law of conservation of mass

$$(\partial/\partial x_k)(\mu u^k) = 0$$

is upheld. (In Einstein's theory one has instead $\Gamma_{i,pq} = -\frac{\mu}{2}\frac{\partial g_{pq}}{\partial x_i}$, provided μ and L_i^k denote the scalar and tensorial *densities*, not scalar and tensor.) But since no theory in which inertia and gravitation are separate entities can explain the universal proportionality of passive gravity and inertial mass, there is no reason why the scalar field σ should be the same as μ (instead one might expect that for a substance *of given chemical constitution* μ and σ are connected by some equation of state $F(\mu, \sigma) = 0$). However, just as Maxwell's L_i^k accounts for the identity of active and passive charge, one can hope in this theory to establish the identity of active and passive gravity by a gravitational energy tensor. For that purpose it is necessary to assume $\sigma u_i u_k$ rather than $\mu u_i u_k + L_{ik}$ as the right member T_{ik} of the field equations $(\tilde{5})$,

$$\tilde{D}_{ik}h = \sigma u_i u_k,$$

and one will try to construct a symmetric tensor G_{ik} which is quadratic in the derivatives $\partial h_{pq}/\partial x_i$ such that the following identity holds:

$$(23) \qquad \partial G_i^k/\partial x_k = (\tfrac{1}{2}\partial h_{ip}/\partial x_q + \tfrac{1}{2}\partial h_{iq}/\partial x_p - \partial h_{pq}/\partial x_i) \cdot \tilde{D}^{pq}h.$$

Then (21) would indeed assume the form of a differential law of conservation of energy and momentum:

$$(24) \qquad (\partial/\partial x_k)(\mu u_i u^k + L_i^k + G_i^k) = 0.$$

There are 16 linearly independent tensors G_{ik} of this sort, and I have checked whether for any linear combination of them a relation like (23) can hold; the result was negative. This applies in particular to the field equations which Birkhoff adopts:

$$\tilde{D}_{ik}h \equiv \Box h_{ik} - \tfrac{1}{2}\Box h \cdot \delta_{ik} = \sigma u_i u_k$$

(and which he interprets in a slightly different manner in terms of a fluid of peculiar nature). It may, therefore, be said that Birkhoff *sacrifices the conservation law of energy and momentum to that of mass.*

That it is possible to develop a theory of dualistic type in which the conservation law for energy-momentum holds is proved by a certain interpretation of the "degenerate Einstein theory" (D) which I had used to illustrate (B): One starts with the field equations of (L) in the normalized form (10 & 11), sets $T_{ik} = \sigma u_i u_k$, *throws away the supplementary conditions* (10) in order to make room for an extra term in the mechanical equations (7) and finally replaces the latter not by (21), but by

$$(\partial/\partial x_k)(\mu u_i u^k + L_i^k) - \frac{\sigma}{2}\frac{\partial h_{pq}}{\partial x_i} u^p u^q = 0.$$

Of course, mass is not conservative in this set-up; one finds instead

$$\partial(\mu u^k)/\partial x_k = (\sigma/6)(\partial h_{pq}/\partial x_r + \partial h_{qr}/\partial x_p + \partial h_{rp}/\partial x_q) u^p u^q u^r.$$

But the conservation laws for energy and momentum (24) hold if one defines the gravitational energy tensor G_{ik} by

$$G_i^k = -H_i^k + \tfrac{1}{2}\delta_i^k \cdot H \qquad (H = H_p^p)$$

where

$$H_{ik} = \frac{1}{2}\frac{\partial h_{pq}}{\partial x_i}\frac{\partial h^{pq}}{\partial x_k} - \frac{1}{4}\frac{\partial h}{\partial x_i}\frac{\partial h}{\partial x_k}.$$

But it is not my intention to propagandize this or any other dualistic theory of gravitation!

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STURMIAN MINIMAL SETS.*

By Gustav A. Hedlund.

1. Introduction. Minimal sets, as defined by G. D. Birkhoff, are of considerable importance in the study of topological dynamics. The simplest example of a minimal set is a periodic motion. Minimal sets which are not periodic motions have been constructed by various devices (cf. e. g., Morse [1]).[1]

If we consider a single transformation and its powers instead of a dynamical flow, a minimal set is defined, essentially as in the case of a flow, to be a closed invariant set which contains no proper subset with the same properties. A simple example of such a set which is not a periodic orbit is a circle with the transformation defined to be a rotation of the circle through an angle which is incommensurable with π. This is an example of a minimal set for which the transformation is regular in the sense of Kerékjártó or, equivalently, has equicontinuous powers. It can be proved that if X is a compact metric space and X is minimal under the homeomorphism $T(X) = X$, then T has equicontinuous powers if and only if T is almost periodic (cf. in this connection, Hartman and Wintner [1]). In this case the minimal set X is necessarily a topological group and the analysis of the structure of X is facilitated by the use of the known properties of topological groups.

It is the purpose of the present paper to define and analyse a class of minimal sets for which the defining transformations do not have equicontinuous powers. The definition of such sets is essentially at hand, for in recent work on symbolic dynamics (cf. Morse and Hedlund [2]) a large class of non-periodic recurrent symbolic trajectories has been defined. By a suitable definition of space and transformation, a recurrent symbolic trajectory defines a recurrent orbit and the closure of a recurrent orbit is a minimal set. However the analysis of the properties of these minimal sets is more involved.

The minimal sets which we consider are totally disconnected compact sets. We show the existence of such sets which contain asymptotic orbits, which are locally almost periodic and which are not only minimal under the defining transformation T but are also minimal under every non-zero power of T.

* Received January 5, 1944.

[1] A list of references will be found at the end of the paper.

With the aid of the sets considered it is possible to throw some light on a conjecture of Birkhoff concerning the homogeneity of recurrent motions (cf. G. D. Birkhoff [2]).

2. Minimal sets, recurrent orbits, almost periodic points. We state several definitions and theorems which are essentially due to G. D. Birkhoff (cf. Birkhoff [1], Chapter VIII). The fact that we are considering a single transformation and its powers rather than the one-parameter group considered by Birkhoff, necessitates a slight change in the definition of a minimal set. We do not impose the condition that a minimal set be connected.

Let X be a compact metric space and let $T(X) = X$ be a homeomorphism.

DEFINITION 2.1. *If x is a point of X, the set $\sum_{-\infty}^{+\infty} T^n(x)$ will be termed the orbit of x and denoted by $O(x)$. The set $\sum_{n=0}^{+\infty} T^n(x)$ will be termed the positive semiorbit of x. The set $\sum_{n=0}^{-\infty} T^n(x)$ will be termed the negative semiorbit of x. The term semiorbit of x will denote either the positive or the negative semiorbit of x.*

DEFINITION 2.2. *The point $y \epsilon X$ is an α-limit point of the orbit $O(x)$ if there exists a sequence of integers $0 > n_1 > n_2 > \cdots$ such that $\lim_{i \to +\infty} T^{n_i}(x) = y$. The set of α-limit points of the orbit $O(x)$ will be denoted by $\alpha(x)$. The point $y \epsilon X$ is an ω-limit point of the orbit $O(x)$ if there exists a sequence of integers $0 < n_1 < n_2 < \cdots$ such that $\lim_{i \to +\infty} T^{n_i}(x) = y$. The set of ω-limit points of the orbit $O(x)$ will be denoted by $\omega(x)$.*

DEFINITION 2.3. *The set $Y \subset X$ is invariant if $T(Y) = Y$.*

The proof of the following theorem is elementary.

THEOREM 2.1. *The sets $\alpha(x)$ and $\omega(x)$ are closed invariant sets.*

DEFINITION 2.4. *The set $Y \subset X$ is a minimal set if Y is non-vacuous, closed and invariant and contains no proper subset with the same properties.*

THEOREM 2.2. *Each of the sets $\alpha(x)$ and $\omega(x)$ contains a minimal set.*

A proof of this for flows is given by Birkhoff (cf. Birkhoff [1], pp. 200-201). Simple proofs of this theorem and the following one are obtained by observing that the property of being a minimal set is inducible and then

making use of the Brouwer Reduction Theorem (cf. in this connection, Kelley [1] and Hall and Kelley [1]).

THEOREM 2.3. *Any non-vacuous closed invariant subset of X contains a minimal set.*

The following theorems are obvious but useful.

THEOREM 2.4. *A necessary condition that a closed invariant subset Y of X be minimal is that each semiorbit of every point of Y be everywhere dense in Y.*

THEOREM 2.5. *A sufficient condition that a closed invariant subset Y of X be minimal is that the orbit of each point of Y be everywhere dense in Y.*

DEFINITION 2.5. *An orbit is recurrent if it lies in a minimal set.*

The following theorem is due to Birkhoff (cf. [1], pp. 199-200).

THEOREM 2.6. *A necessary and sufficient condition that the orbit $O(x)$ be recurrent is that corresponding to $\epsilon > 0$ there exist an integer $N > 0$ such that $O(x)$ is contained in the ϵ-neighborhood of the set*

$$\{T^{i+1}(x), T^{i+2}(x), \cdots, T^{i+N}(x)\}$$

for each integral i.

DEFINITION 2.6. *The sequence of integers*

$$\cdots < n_{-1} < n_0 < n_1 < n_2 < \cdots \tag{2.1}$$

is relatively dense if there exists an integer N such that $n_i - n_{i-1} \leqq N$ for all i. The integer N is termed an inclusion integer of the set (2.1).

DEFINITION 2.7. *The point $x \epsilon X$ is almost periodic if, corresponding to $\epsilon > 0$, there exists a relatively dense sequence of integers* (2.1) *such that*

$$d(x, T^{n_i}(x)) < \epsilon, \qquad (i = 0, \pm 1, \cdots).$$

The following theorem has been proved by Hall and Kelley ([1], p. 628).

THEOREM 2.7. *A necessary and sufficient condition that the point $x \epsilon X$ be almost periodic is that the orbit $O(x)$ be recurrent.*

The sufficiency of the condition is an obvious consequence of Theorem 2.6.

In view of Theorem 2.7, there is a superfluity of terms and it would appear desirable to drop one of the terms *almost periodic* or *recurrent*. Since

the term *recurrent* appears frequently in the literature, there is reason to retain recurrent with its present significance. But the term *almost periodic*, as defined in Definition 2. 7, is much more suggestive of the property defined and the author feels strongly that this definition should be retained. Then the term *recurrent* could be employed to replace the numerous confusing terms which have been used for the property defined by Definition 2. 7 without the restriction that the sequence of integers (2. 1) be relatively dense. This property has been associated with the expressions *stable in the sense of Poisson* (Poincaré), *pseudo-recurrent* (Birkhoff and Smith), *pervasive* (Cherry), and *almost periodic* (Ayres). The term *recurrent* seems more appropriate than any of these.

3. The space of symbolic elements and the transformation S. Let C be a finite set of μ distinct symbols termed the *generating symbols* and let A denote a sequence of the form

$$\text{(A)} \qquad \cdots a_{-1}a_0a_1a_2\cdots$$

where a_n represents a generating symbol. The generating symbol represented by a_n is termed the *value* of a_n. As in SDI, **2**, we term A an I-trajectory. The I-trajectory

$$\text{(B)} \qquad \cdots b_{-1}b_0b_1b_2\cdots$$

is *identical* with A if and only if a_n and b_n have the same value for each integral value of n. In this case we write $A \equiv B$. If there exists an integer r such that a_n and b_{n+r} have the same value for each integral value of n, A and B are said to be *similar*. The class of I-trajectories similar to any one I-trajectory is termed a *symbolic trajectory*. The symbolic trajectory defined by such a class is said to be *represented* by any one of the I-trajectories in the class.

The pair consisting of the I-trajectory A and its r-th symbol a_r is termed an *I-element* and is denoted by $e(r, A)$. The I-elements $e(r, A)$ and $e(s, B)$ are identical if and only if $A \equiv B$ and $r = s$. The I-elements $e(r, A)$ and $e(s, B)$ are *similar* if and only if a_{r+n} and b_{s+n} have the same value for all integral values of n. The class of all I-elements similar to a given I-element $e(r, A)$ is termed a *symbolic element e*. Two symbolic elements are equal if and only if the classes of I-elements defining them are identical. If $e(r, A)$ is a member of the class of symbolic I-elements defining e, e is said to be *based* on A and represented by $e(r, A)$. It is evident that corresponding to any element e there exists an I-trajectory A such that e is represented by $e(0, A)$ and this representation is unique.

Let e_1 and e_2 be symbolic elements and let them be represented by $e(0, A)$ and $e(0, B)$ respectively. If $e_1 = e_2$, we define the distance between e_1 and e_2 to be zero. If this is not the case, let m be the greatest integer such that the values of a_n and b_n are the same for $n = 0, \pm 1, \pm 2, \cdots, \pm m$. The distance between e_1 and e_2 is then defined to be $1/(m+1)$. It is easily proved (cf. SDI, p. 819) that this distance satisfies the usual metric axioms. It is also easy to prove that if $E(C)$ denotes the space of symbolic elements based on the symbolic trajectories which can be constructed by use of the given class C of generating symbols, then $E(C)$ is compact, perfect and totally disconnected. Thus $E(C)$ is a homeomorph of the Cantor discontinuum (cf. SDI, p. 820).

Let e be an arbitrary point (symbolic element) of the space $E(C)$ and let e be represented by the symbolic I-element $e(r, A)$ where A is given by

$$\text{(A)} \qquad \cdots a_{-1}a_0a_1a_2 \cdots .$$

Let $S(e)$ be the symbolic element represented by $e(r+1, A)$, or, equivalently, by $e(r, B)$ where B is given by

$$\text{(B)} \qquad \cdots b_{-1}b_0b_1b_2 \cdots$$

and the values of b_n and a_{n+1} are the same for all integral values of n. The element $S(e)$ is evidently independent of the choice of the I-element $e(r, A)$ representing e.

THEOREM 3.1. *The transformation $e \to S(e)$ is a one-to-one continuous transformation of $E(C)$ onto $E(C)$.*

If the distance between the elements e_1 and e_2 is $1/(m+1)$, it follows that the distance between $S(e_1)$ and $S(e_2)$ is not greater than $1/m$. We infer that the transformation S is continuous.

If e is represented by $e(r, A)$, let $S^*(e)$ denote the symbolic element represented by $e(r-1, A)$. Again, $S^*(e)$ is independent of the choice of I-element representing e and is continuous. It follows from the definition of S and S^* that $S^*S(e) = SS^*(e) = e$. We infer that S is a one-to-one transformation of $E(C)$ onto $E(C)$ and that S^* is the inverse of S.

If e is represented by $e(r, A)$, then $S^n(e)$ is the element represented by $e(r+n, A)$. Thus the orbit of e under S and its powers is simply the set of elements based on A.

Let e be represented by $e(r, A)$ and let $S^n(e)$ be represented by $e(s, B)$. Then, since $S^n(e)$ is also represented by $e(r+n, A)$, we infer from the similarity of $e(s, B)$ and $e(r+n, A)$ that the I-trajectories A and B are

similar. Thus, if $O(e)$ is an orbit in $E(C)$, the class of I-trajectories on which the points of $O(e)$ can be based is a class of similar I-trajectories. Conversely, if A and B are similar I-trajectories and e_1 and e_2 are points of $E(C)$ based on $e(r, A)$ and $e(s, B)$, respectively, it follows from the similarity of A and B that an integer k exists such that $e(r + k, A)$ and $e(S, B)$ are similar, and consequently, e_2 is on the orbit of e_1. Thus the points of $E(C)$ based on similar I-trajectories lie on the same orbit. Since the class of I-trajectories similar to an I-trajectory defines a symbolic trajectory, it follows that the orbits in $E(C)$ and the symbolic trajectories constructed from the class C of symbols are in one-to-one correspondence.

4. Minimal sets in $E(C)$. Let e be a point of $E(C)$, let $O(e)$ be the orbit of e under the transformation S and let h denote the corresponding symbolic trajectory. The problem of constructing an orbit with given properties is now equivalent to constructing a symbolic trajectory with specified properties. In particular, a necessary and sufficient condition that $O(e)$ be periodic is that h be periodic (cf. SDI, p. 824). It is easily proved that the periodic orbits form a set which is everywhere dense in $E(C)$.

The symbolic trajectory h is said to be *almost periodic* if, corresponding to an arbitrary integer n, there exists an integer m such that all the n-blocks which appear in h appear in each m-block of h. With the aid of Theorem 2. 7, it is not difficult to prove that the orbit $O(e)$ is almost periodic if and only if h is almost periodic. To construct a minimal set which is not a periodic orbit, it is sufficient to construct a symbolic trajectory which is almost periodic but not periodic. The minimal set is the closure of the corresponding orbit $O(e)$.

Almost periodic symbolic trajectories which are not periodic, have been constructed by various devices. The first example of such a symbolic trajectory was constructed by Morse (cf. Morse [1]). Methods for constructing examples of quite different type have been defined by Birkhoff (cf. Birkhoff [3], p. 22). The Sturmian symbolic trajectories, defined and analysed in some detail by Morse and Hedlund (cf. SDII), are non-periodic whenever the frequency is irrational and are of the same type as those considered by Birkhoff.

All of these symbolic trajectories define almost periodic orbits in the space $E(C)$, and the closures of these orbits define minimal sets. In particular, the minimal sets obtained by this procedure from Sturmian symbolic trajectories will be termed *Sturmian minimal sets.* The object of the present paper is to study the properties of Sturmian minimal sets and the properties

of the transformation S and its powers, considered as acting on one of these minimal sets. In order to do this, it appears to be desirable to give a new mechanical construction of these symbolic trajectories (cf. SDII, p. 13).

5. Sturmian minimal sets. Let β be a positive irrational number and let I_i denote the interval

$$i(1+\beta) \leqq x < (i+1)(1+\beta), \qquad (i = 0, \pm 1, \pm 2, \cdots),$$

of the real x-axis. Let the interval I_i be divided into the two intervals

$$B_i, i(1+\beta) \leqq x < i(1+\beta)+1; \quad A_i, i(1+\beta)+1 \leqq x < (i+1)(1+\beta).$$

We term the first of these a *b-interval* and the second an *a-interval.* Corresponding to an arbitrary real number c we introduce the set of points

$$(5.1) \qquad \cdots, c - 2\beta,\ c - \beta,\ c,\ c+\beta,\ c+2\beta, \cdots.$$

Let C_2 be the class of two symbols a and b and let A be the symbolic I-trajectory

$$(\mathrm{A}) \qquad \cdots a_{-2}a_{-1}a_0a_1a_2 \cdots$$

where the value of a_i is a or b according as $c + i\beta$ is in an a-interval or in a b-interval. We denote by $e(c,\beta)$ the symbolic element or point of E which is represented by the symbolic I-element $e(0,A)$. It follows that $S^n(e(c,\beta)) = e(c+n\beta,\beta)$. We denote the symbolic trajectory defined by A by $t(c,\beta)$.

If we denote by I'_i, B'_i and A'_i intervals with the same end points as I_i, B_i and A_i, respectively, but which are open on the left and closed on the right, and otherwise use the same procedure, we obtain a symbolic element $e'(c,\beta)$ and a symbolic trajectory $t'(c,\beta)$.

THEOREM 5.1. *If* $c \equiv d$, *mod* $(1+\beta)$, *then* $e(c,\beta) = e(d,\beta)$, $t(c,\beta) = t(d,\beta)$, $e'(c,\beta) = e'(d,\beta)$, *and* $t'(c,\beta) = t'(d,\beta)$.

For if $c \equiv d$, mod $(1+\beta)$, there exists an integral value of k such that $c = d + k(1+\beta)$. But the translation $x' = x + k(1+\beta)$ transforms a-intervals into a-intervals and b-intervals into b-intervals. It follows that $c + n\beta$ is in an a-interval or in a b-interval according as $d + n\beta$ is in an a-interval or in a b-interval. This implies the statement of the theorem.

THEOREM 5.2. *If* $e(c,\beta) = e(d,\beta)$ *or* $e'(c,\beta) = e'(d,\beta)$, *then* $c \equiv d$. *mod* $(1+\beta)$.

Since β is irrational, β and $1+\beta$ are incommensurable, and it is well known that in this case, the set of points (5.1), reduced mod $(1+\beta)$ so that they lie in the interval I_1, $0 \leqq x < 1+\beta$, is everywhere dense in this interval.

In virtue of Theorem 5.1 we can assume that both c and d lie in I_1. Assuming that the statement of the theorem is not true, we infer that c and d are not identical points of I_1. The point c' in I_1 can be so chosen that it lies in the interior of a b-interval, while the point $c' + (d - c) = d'$ lies in the interior of an a-interval. Given $\epsilon > 0$, there exists an integer n such that when $c + n(1+\beta)$ is reduced mod $(1+\beta)$ it lies within distance ϵ of c'. But then $d + n(1+\beta)$ can be reduced mod $(1+\beta)$ so that it lies within distance ϵ of d'. If ϵ is chosen sufficiently small, it follows that $c + n\beta$ lies in a b-interval, whereas $d + n\beta$ lies in an a-interval, contrary to the hypothesis of the theorem that either $e(c,\beta) = e(d,\beta)$ or $e'(c,\beta) = e'(d,\beta)$.

THEOREM 5.3. *$e(c,\beta) = e'(d,\beta)$ if and only if $c \equiv d$, mod $(1+\beta)$, and $c \not\equiv m$, mod β, where m is an integer.*

Let us assume that $c \equiv d$, mod $(1+\beta)$, and that $c \not\equiv m$, mod β. It follows from Theorem 5.1 that $e(c,\beta) = e(d,\beta)$. Since $c \not\equiv m$, mod β, and $c \equiv d$, mod $(1+\beta)$, we infer that $d \not\equiv m$, mod β, and hence no one of the points

$$\cdots, d - \beta,\ d,\ d + \beta,\ d + 2\beta, \cdots$$

is an endpoint of an a-interval or of a b-interval. But since the intervals A_i and A'_i have the same interior points, and likewise for B_i and B'_i, it follows that $e(d,\beta) = e'(d,\beta)$. Since $e(c,\beta) = e(d,\beta)$, we obtain $e(c,\beta) = e'(d,\beta)$.

Now let us assume that $e(c,\beta)' = e'(d,\beta)$. In view of Theorem 5.1, there exist points $\bar{c}$ and $\bar{d}$ in I_1 such that $c \equiv \bar{c}$, mod $(1+\beta)$, $d \equiv \bar{d}$, mod $(1+\beta)$, and hence $e(c,\beta) = e(\bar{c},\beta)$ and $e'(d,\beta) = e'(\bar{d},\beta)$. If $\bar{c}$ and $\bar{d}$ are not identical it follows by the method used to prove Theorem 5.2 that $e(\bar{c},\beta)$ and $e'(\bar{d},\beta)$ are not identical. But since the identity of these is implied by the hypothesis that $e(c,\beta) = e'(d,\beta)$, we infer that $\bar{c} = \bar{d}$, and hence, in particular, $c \equiv d$, mod $(1+\beta)$. Suppose that $c \equiv m$, mod β, where m is an integer. Then there exists an integral k such that $c = m + k\beta$. Since $c \equiv \bar{c}$, mod $(1+\beta)$, there exists an integral j such that $\bar{c} = c + j(1+\beta)$. By substitution we obtain

$$\bar{c} = m + k\beta + j(1+\beta) = (k - m)\beta + (m + j)(1+\beta)$$

or

$$\bar{c} + (m - k)\beta = (m + j)(1 + \beta).$$

But this implies that the point $\bar{c} + (m - k)\beta$ lies in the interval B_{m+j} and the corresponding symbol in $e(\bar{c}, \beta)$ is b. Since $\bar{c} = \bar{d}$, the point $\bar{d} + (m - k)\beta$ coincides with $\bar{c} + (m - k)\beta$, lies in A'_{m+j-1}, and the corresponding symbol in $e'(\bar{d}, \beta)$ is a. This implies that $e(\bar{c}, \beta) \neq e'(\bar{d}, \beta)$, and consequently $e(c, \beta) \neq e'(d, \beta)$, contrary to hypothesis. We infer that $c \not\equiv m, \bmod \beta$, and the proof of the theorem is complete.

THEOREM 5.4. *If $c \not\equiv m, \bmod \beta$, m an integer, then*

$$\lim_{x \to c} e(x, \beta) = \lim_{x \to c} e'(x, \beta) = e(c, \beta) = e'(c, \beta).$$

If $c \equiv m, \bmod \beta$, m an integer, then

$$\lim_{x \to c+} e(x, \beta) = \lim_{x \to c+} e'(x, \beta) = e(c, \beta)$$

and

$$\lim_{x \to c-} e(x, \beta) = \lim_{x \to c-} e'(x, \beta) = e'(c, \beta).$$

For if $c \not\equiv m, \bmod \beta$, no one of the points

$$\cdots, c - \beta,\ c,\ c + \beta,\ c + 2\beta, \cdots$$

is an endpoint of one of the intervals A_i, B_i, A'_i or B'_i. Given any positive integer k, there exists an $\epsilon > 0$ such that if $|x - c| < \epsilon$ and $|n| \leqq k$, then $c + n\beta$ and $x + n\beta$ lie in the same a-interval or b-interval. It follows that the distance between $e(x, \beta)$ and $e(c, \beta)$ or between $e'(x, \beta)$ and $e'(x, \beta)$ is less than $1/(k + 1)$. This implies the first statement of the theorem.

To prove the second statement of the theorem we consider first the case $c = 0$. Then no one of the points

$$\cdots, c - 2\beta,\ c - \beta,\ c + \beta,\ c + 2\beta, \cdots$$

is an endpoint of one of the intervals A_i, B_i, A'_i or B'_i and, as before, if $\epsilon > 0$ is chosen sufficiently small, $c + n\beta$ and $x + n\beta$, $0 < n \leqq k$, lie in the same a-interval or b-interval provided $0 < x < \epsilon$. But if $\epsilon < 1$, x and $c = 0$ lie in the same b-interval B_1. Thus

$$\lim_{x \to 0+} e(x, \beta) = \lim_{x \to 0+} e'(x, \beta) = e(0, \beta).$$

Similarly it can be proved that

$$\lim_{x\to 0-} e(x,\beta) = \lim_{x\to 0-} e'(x,\beta) = e'(0,\beta).$$

We infer the validity of the theorem in the case $c = p(1+\beta)$, p integral, from the case $c = 0$ by use of Theorem 5.1.

If $c \equiv m$, mod β, m an integer, there exists an integral value of q such that $c = m + q\beta$, and hence $c + (m - q)\beta = m(1+\beta)$. From the cases previously considered,

$$\lim_{x\to m(1+\beta)+} e(x,\beta) = \lim_{x\to m(1+\beta)+} e'(x,\beta) = e(m(1+\beta),\beta)$$

and

$$\lim_{x\to m(1+\beta)-} e(x,\beta) = \lim_{x\to m(1+\beta)-} e'(x,\beta) = e'(m(1+\beta),\beta).$$

But

$$S^{q-m}[e(x,\beta)] = e(x + [q-m]\beta,\beta)$$

and

$$S^{q-m}[e'(x + [q-m]\beta,\beta) = e'(x + [q-m]\beta,\beta).$$

Since S^{q-m} is a continuous transformation, we infer the validity of the theorem in the general case.

Let $M(\beta)$ denote the set of points $e(c,\beta)$ and $e'(c,\beta)$ where c is an arbitrary real number. The set $M(\beta)$ is then a subset of a metric space and hence is a metric space.

Let e be any point of $M(\beta)$. Then there exists a number c such that either $e(c,\beta)$ or $e'(c,\beta)$ is equal to e. We term such a number c a *real number corresponding to e.* It follows from Theorem 5.2 that there are infinitely many real numbers corresponding to e, but any two of them are congruent mod $(1+\beta)$.

THEOREM 5.5. *Let $e_1, e_2, \cdots$ be a sequence of points of $M(\beta)$ such that $\lim_{n\to\infty} e_n = e$. Let $c_1, c_2, \cdots$ and c be real numbers corresponding to $e_1, e_2, \cdots$ and e, respectively. Then there exists a sequence*

$$c'_1, c'_2, \cdots, c'_i \equiv c_i, \text{mod } (1+\beta), \qquad (i = 1, 2, \cdots),$$

such that if $e = e(c,\beta) = e'(c,\beta)$, then $\lim_{n\to\infty} c'_n = c$; if $e = e(c,\beta) \neq e'(c,\beta)$, then $\lim_{n\to\infty} c'_n = c+$; and if $e = e'(c,\beta) \neq e(c,\beta)$, then $\lim_{n\to\infty} c'_n = c-$.

We consider first the case $c \not\equiv m$, mod β, m an integer. This is the case when $e(c,\beta) = e'(c,\beta)$. Let I denote the particular interval I_i in which c lies. Then c is not an endpoint of I. The sequence $c'_1, c'_2, \cdots$ can be so chosen that $c'_i \equiv c_i$, mod $(1+\beta)$, $i = 1, 2, \cdots$, and c'_i is in I. If the sequence $c'_1, c'_2, \cdots$ does not converge to c, this sequence contains a con-

vergent subsequence $c'_{n_1}, c'_{n_2}, \cdots$ such that $\lim_{k\to\infty} c'_{n_k} = \bar{c} \neq c$ and $\bar{c}$ lies in the interval $i(1+\beta) \leqq x \leqq (i+1)(1+\beta)$, which is the closure of I. It follows from Theorem 5.4 that the sequence $e_1, e_2, \cdots$ must converge either to $e(\bar{c},\beta)$ or to $e'(\bar{c},\beta)$. But since c is not an endpoint of I, it follows from Theorem 5.2 that $e(\bar{c},\beta) \neq e(c,\beta) = e$ and $e'(\bar{c},\beta) \neq e'(c,\beta) = e$. Thus the sequence $c'_1, c'_2, \cdots$ must converge to c.

Now let us consider the case $e = e(c,\beta) \neq e'(c,\beta)$ and thus $c \equiv m$, mod β, m an integer. We choose the sequence $c'_1, c'_2 \cdots$ as in the first case considered. If $\lim_{n\to\infty} c'_n = c$, then it follows from Theorem 5.4 that $\lim_{n\to\infty} c'_n = c+$. If the sequence $c'_1, c'_2, \cdots$ does not converge to c, we choose, as before, a subsequence converging to $\bar{c} \neq c$. Again the sequence $e_1, e_2, \cdots$ must converge either to $e(\bar{c},\beta)$ or to $e'(\bar{c},\beta)$, and the latter is impossible according to Theorem 5.3. But then $\lim_{n\to\infty} e_n = e(\bar{c},\beta)$, we have $e(c,\beta) = e(\bar{c},\beta)$, and hence, according to Theorem 5.2, $c \equiv \bar{c}$, mod $(1+\beta)$. This would imply that c is the left end and $\bar{c}$ is the right end point of I. But then, according to Theorem 5.4, $\lim_{i\to\infty} e(c'_{n_i},\beta) = e'(\bar{c},\beta) \neq e(c,\beta)$. Thus we must have $c = \bar{c}$ and the second assertion of the theorem is proved.

The final case, when $e = e'(c,\beta) \neq e(c,\beta)$, can be treated similarly.

THEOREM 5.6. *The set $M(\beta)$ is minimal under the transformation S.*

To prove that the set $M(\beta)$ is minimal under S it is sufficient to prove that $M(\beta)$ is invariant under S, $M(\beta)$ is a closed subset of $E(C_2)$ and the orbit of each point of $M(\beta)$ is everywhere dense in $M(\beta)$.

Since $S^n[e(c,\beta)] = e(c+n\beta,\beta)$ and $S^n[e'(c,\beta)] = e'(c+n\beta,\beta)$, the orbit of each point of $M(\beta)$ lies in $M(\beta)$ and the set $M(\beta)$ is invariant under S.

To prove that $M(\beta)$ is a closed subset of $E(C_2)$ it is sufficient to prove that any infinite sequence of points of $M(\beta)$ contains a subsequence which converges to a point of $M(\beta)$. Any infinite sequence of points of $M(\beta)$ must contain either an infinite subsequence of the form

$$(5.2) \qquad e(c_1,\beta),\quad e(c_2,\beta),\quad e(c_3,\beta), \cdots$$

or an infinite subsequence of the form

$$e'(c_1,\beta),\quad e'(c_2,\beta),\quad e'(c_3,\beta), \cdots.$$

We can assume without loss of generality that a subsequence of the type (5.2) occurs. In view of Theorem 5.1, we can assume that the points

$c_1, c_2, c_3, \cdots$ all lie in the interval $0 \leqq x \leqq 1 + \beta$. But then the sequence $c_1, c_2, c_3, \cdots$ contains a subsequence which converges to a point c such that $0 \leqq c \leqq 1 + \beta$. We can assume that the sequence $c_1, c_2, c_3, \cdots$ has this property. But then this sequence contains a subsequence which converges to c from the left or else a subsequence which converges to c from the right. Assuming that the first property holds for the sequence $c_1, c_2, \cdots$, that is, $\lim_{i \to \infty} c_i = c-$, we infer from Theorem 5.4 that $\lim_{i \to \infty} e(c_i, \beta) = e'(c, \beta)$. In the alternative case, $\lim_{i \to \infty} e(c_i, \beta) = e(c, \beta)$, and the desired result is proved.

Since $S^n[e(c, \beta)] = e(c + n\beta, \beta)$, $S^n[e'(c, \beta)] = e'(c + n\beta, \beta)$ and the points

$$\cdots, c - \beta, \quad c, \quad c + \beta, \quad c + 2\beta, \cdots$$

when reduced mod $(1 + \beta)$ form a set which is everywhere dense in the interval I_1, it follows from Theorems 5.1 and 5.4 that the orbit of any point of $M(\beta)$ is everywhere dense in $M(\beta)$.

The proof that $M(\beta)$ is a minimal set is complete.

6. Properties of the minimal set $M(\beta)$. The set $M(\beta)$, which has been shown to be minimal under the transformation S, is a subset of the totally disconnected set $E(C_2)$ and hence is itself totally disconnected. It follows from Theorem 5.2 that $M(\beta)$ is not a finite set and thus is not a periodic orbit. Since $M(\beta)$ contains more than one orbit and each orbit is everywhere dense in $M(\beta)$, each point of $M(\beta)$ is a limit point of points of $M(\beta)$ and $M(\beta)$ is dense-in-itself. Since $M(\beta)$ is a closed subset of a compact space $E(C_2)$, $M(\beta)$ is compact. Thus we can state the following theorem.

THEOREM 6.1. *The minimal set $M(\beta)$ is compact, perfect and totally disconnected.*

Let $d(e, e^*)$ denote the distance between the points e and e^* of $E(C_2)$. Let $S^n(e) = e_n$ and $S^n(e^*) = e^*_n$, $n = 0, \pm 1, \pm 2, \cdots$. The orbits of e and of e^* will be said to be *positively asymptotic* (*negatively asymptotic*) if $\lim_{n \to +\infty} d(e_n, e^*_n) = 0$ $(\lim_{n \to -\infty} d(e_n, e^*_n) = 0)$. The orbits will be said to be *doubly asymptotic* if they are both positively and negatively asymptotic.

THEOREM 6.2. *The minimal set $M(\beta)$ contains a pair of doubly asymptotic orbits.*

Let us consider the distinct points or symbolic elements $e(0, \beta)$ and $e'(0, \beta)$. The points

$$\cdots, -2\beta, -\beta, \beta, 2\beta, 3\beta, \cdots$$

are all interior points of a-intervals or of b-intervals and consequently the symbols of $e(0,\beta)$ and $e'(0,\beta)$ which correspond to any one of these points are identical. It follows that

$$d[S^n e(0,\beta),\ S^n e'(0,\beta)] = d[e(n\beta,\beta),\ e'(n\beta,\beta)] = 1/(n+1).$$

Thus

$$\lim_{n\to\pm\infty} d[S^n e(0,\beta),\ S^n e'(0,\beta)] = 0,$$

and the orbits $e(0,\beta)$ and $e'(0,\beta)$ are asymptotic.

Let X be a metric space and let $T(X) = X$ be a homeomorphism of X onto X. The homeomorphism $R(X) = X$ of X onto X is said to be *orbit preserving* with respect to T if $R(T^n(x)) = T^n(R(x))$ for all points x of X and all integral n. A question raised by Birkhoff [3] concerning continuous flows suggests the following question concerning minimal sets in the case of a single transformation. Given a minimal set and a pair of points of the set, does there necessarily exist an orbit preserving homeomorphism of the minimal set onto itself transforming one of these points into the other? With the aid of Theorem 6.2 we can conclude that the answer is in the negative.

For suppose that $R(M(\beta)) = M(\beta)$ is an homeomorphism of $M(\beta)$ onto $M(\beta)$ such that R is orbit preserving with respect to S and such that $R[e(0,\beta)] = e'(0,\beta)$. Since $e(0,\beta) \neq e'(0,\beta)$, R is not the identity. There are no points of $M(\beta)$ which are fixed under R. For if $R(e) = e$, then all points on the orbit of e are fixed points under R and since the points of any orbit are everywhere dense in $M(\beta)$ it would follow that the fixed points of R would be everywhere dense in $M(\beta)$. This would imply that R is the identity, which is not the case. Since there are no points of $M(\beta)$ which are fixed under R and since $M(\beta)$ is compact, there exists a $\delta > 0$ such that $d[R(e), e] > \delta$ for all e in $M(\beta)$. But then the orbits of $e(0,\beta)$ and $e'(0,\beta)$ could not be doubly asymptotic, contrary to Theorem 6.2. Thus the orbit preserving transformation R cannot exist.

If, however, the compact metric space X is minimal under the regular homeomorphism $T(X) = X$, then it can be proved that given any two points of X, there exists an orbit preserving homeomorphism of X onto X transforming one of these points into the other. As to whether the converse of this statement is true or not appears to be an unsolved problem.

The homeomorphism $T(X) = X$ of the metric space X onto X is said to be *almost periodic* if, corresponding to $\epsilon > 0$, there exists a relatively dense sequence of integers

$$(6.1) \qquad \cdots < n_{-1} < n_0 < n_1 < n_2 < \cdots$$

such that $d[T^{n_i}(x), x] < \epsilon$ for all x in X and all integral values of i. We note that the transformation S is not almost periodic on the minimal set $M(\beta)$. For if S were almost periodic on $M(\beta)$, let us choose $\epsilon = 1/3$ and let (6.1) denote the corresponding relatively dense set of integers. If $e = e(0, \beta)$ and $e' = e'(0, \beta)$, then $d(e, e') = 1$ and since $d[S^{n_i}(e), e] < 1/3$ and $d[S^{n_i}(e'), e'] < 1/3$, we infer that $d[S^{n_i}(e), S^{n_i}(e')] > 1/3$ for all i. It follows that the orbits of e and e' could not be doubly asymptotic, contrary to Theorem 6.2. Thus S cannot be almost periodic on $M(\beta)$.

The homeomorphism $T(X) = X$ of the metric space X onto X will be said to be *locally almost periodic* on X if, given any point x of X and any neighborhood $U(x)$ of x, there exists a neighborhood $V(x)$ of x and a relatively dense sequence of integers (6.1) such that $T^{n_i}[V(x)] \subset U(x)$ for $i = 0, \pm 1, \pm 2, \cdots$. We prove that $S[M(\beta)] = M(\beta)$ is locally almost periodic. In order to do this we introduce another but equivalent topology in the space $M(\beta)$.

Let e be any point of $M(\beta)$. Then there exists a constant c such that either $e = e(c, \beta)$ or $e = e'(c, \beta)$. We consider first the case $e = e(c, \beta)$ and $c \not\equiv m, \bmod \beta$, m an integer. Then, according to Theorem 5.2, $e(c, \beta) = e'(c, \beta)$ and in this case we define a neighborhood U of e to be the set of all $e(x, \beta)$ and $e'(x, \beta)$ such that $| x - c | < \delta$, where δ is a positive number. If $e = e(c, \beta)$ and $c \equiv m, \bmod \beta$, m an integer, we define a neighborhood of e to be the set of all $e(x, \beta)$ and $e'(x, \beta)$ such that $c \leqq x < c + \delta$, where δ is a positive number. If $e = e'(c, \beta)$ and $c \equiv m, \bmod \beta$, m an integer, we define a neighborhood of e to be the set $e(x, \beta)$ and $e'(x, \beta)$ such that $c - \delta < x \leqq c$, where δ is a positive number. It is not difficult to verify that these neighborhoods satisfy the Hausdorff axioms as well as the conditions of regularity and separability.

Since $M(\beta)$ is a subset of the metric space $E(C_2)$, $M(\beta)$ is a metric space. The equivalence of the two topologies defined by the neighborhoods U and the metric in $M(\beta)$ is a simple consequence of Theorems 4.4 and 4.5.

THEOREM 6.3. *The transformation S is locally almost periodic on the minimal set $M(\beta)$.*

Let e be an arbitrary point of $M(\beta)$. Then there exists a real number c such that either $e = e(c, \beta)$ or $e = e'(c, \beta)$. We consider first the case when $e = e(c, \beta)$ and $c \not\equiv m$, $\bmod \beta$, m an integer. Then $e = e(c, \beta) = e'(c, \beta)$. Let U be the neighborhood of e defined by the set $e(x, \beta)$, $| x - c | < \delta > 0$,

and let V denote the neighborhood of e defined by the set $e(x,\beta)$, $|x-c|<\delta/2$. Evidently $V\subset U$. Since $M(\beta)$ is a minimal set, each point of it is almost periodic and there exists a relatively dense sequence (6.1) of integers such that $S^{n_i}(e)\,\epsilon\,V$, $i=0,\pm 1,\pm 2,\cdots$. But

$$S^{n_i}(e)=S^{n_i}[e(c,\beta)]=e(c+n_i\beta,\beta)$$

and hence $c+n_i\beta\equiv d_i$, mod $(1+\beta)$, where $|d_i-c|<\epsilon/2$. Now $S^{n_i}(V)$ is defined by the set $e(x+n_i\beta,\beta)$, $|x-c|<\delta/2$. But then for each such x and integer i, there exists a $y(x,i)$ such that $x+n_i\beta\equiv y(x,i)$, mod $(1+\beta)$, where $|y(x,i)-d_i|<\delta/2$. It follows that $|y(x,i)-c|<\delta$ and hence $e(x+n_i\beta,\beta)\,\epsilon\,U$. Thus $S^{n_i}(V)\subset U$, $i=0,\pm 1,\pm 2,\cdots$, and the theorem is proved in the case under consideration.

The proofs in the other cases are similar and will be omitted.

Let the compact metric space X be minimal under the homeomorphism $T(X)=X$ of X onto itself. The minimal set X will be said to be *powerfully minimal* if X is minimal under all powers of T other than $T^0=$ the identity. If the number of components of X is finite and greater than one, X cannot be powerfully minimal. The following theorem shows, however, that a set may have infinitely many components and yet be powerfully minimal.

THEOREM 6.4. *The set $M(\beta)$, which is minimal under the transformation S, is powerfully minimal under S.*

For let k be any non-zero integer and let $T=S^k$. If e denotes any point of $M(\beta)$, let c be a real value corresponding to e. Then we have either $e=e(c,\beta)$ or $e=e'(c,\beta)$. The arguments in the two cases are similar, so we shall assume that $e=e(c,\beta)$. Then the points of the orbit of e under T and its powers are the points

$$\cdots, e(c-k\beta,\beta),\ e(c,\beta),\ e(c+k\beta,\beta)\cdots.$$

But $k\beta$ and $1+\beta$ are incommensurable, so that the points

$$\cdots, c-k\beta, c,\ c+k\beta,\ c+2k\beta,\cdots$$

reduced mod $(1+\beta)$ to the interval I_1 are everywhere dense in this interval. It follows that the orbit of e under T and its powers is everywhere dense in $M(\beta)$. But this implies that $M(\beta)$ is minimal under T and hence powerfully minimal under S.

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BIBLIOGRAPHY

Birkhoff, G. D.

1. "Dynamical Systems," *American Mathematical Society Colloquium Publications*, vol. 9 (New York, 1927).

2. "Some unsolved problems of theoretical dynamics," *Science*, vol. 94 (1941), pp. 598-600.

3. "Sur le problème restreint des trois corps (Second Mémoire)," *Annali della R. Scuola Normale Superiore di Pisa* (Serie II), t. 5 (1936), pp. 1-42.

Hall, D. W. and J. L. Kelley

1. "Periodic types of transformations," *Duke Mathematical Journal*, vol. 8 (1941), pp. 625-630.

Hartman, P. and A. Wintner

1. "Integrability in the large and dynamical stability," *American Journal of Mathematics*, vol. 65 (1943), pp. 273-278.

Kelley, J. L.

1. "Fixed sets under homeomorphisms," *Duke Mathematical Journal*, vol. 5 (1939), pp. 535-537.

Morse, M.

1. "Recurrent geodesics on a surface of negative curvature," *Transactions of the American Mathematical Society*, vol. 22 (1921), pp. 33-51.

Morse, M. and G. A. Hedlund

1. "Symbolic Dynamics," *American Journal of Mathematics*, vol. 60 (1938), pp. 815-866. This paper is referred to as SDI.

2. "Symbolic Dynamics II. Sturmian Trajectories," *American Journal of Mathematics*, vol. 62 (1940), pp. 1-42. This paper is referred to as SDII.

VARIETY CONGRUENCES OF ORDER ONE IN n-DIMENSIONAL SPACE.*

By Edwin J. Purcell.

A variety congruence of order one in $[n]$ *is an algebraic* ∞^k*-system of varieties, each of dimension* $n - k$ *and order* h*, in n-dimensional projective space, such that through a generic point of* $[n]$ *one and only one* V^h_{n-k} *of the system passes*, (k any positive integer not greater than n, and h any positive integer).

A generic point P of the ambient $[n]$ determines uniquely a V^h_{n-k} through P, and this same V^h_{n-k} is likewise determined by any other non-fundamental point on it.

This paper treats variety congruences of order one in $[n]$, where the generic variety may be of any dimension less than n and of any order whatever. The congruences are classified and their fundamental loci discussed. Curve congruences of order one in [3] are examined in more detail.

When $h = 1$, the generic variety of the congruence in $[n]$ is a flat space.[1]

When $n = k$, and h is any positive integer, the generic variety of the congruence is a group of h points having the property that any point of the group determines the remaining $h - 1$ points of the same group.

When $n = k$ and $h = 2$, each congruence establishes a Cremona involution in $[n]$. Any point of $[n]$ and its correspondent in the involution form a V_0^2 of the congruence, and conversely. These Cremona involutions will be treated in a separate paper.

The results of very many writers on Cremona transformations, Cremona involutions, $(1.m)$ correspondences, and line or curve congruences of order one, can be obtained by specializing the present paper.[2]

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* Received July 24, 1943.

[1] E. J. Purcell, "Flat space congruences of order one in $[n]$," *Transactions of the American Mathematical Society*, vol. 54 (1943), pp. 57-69.

[2] For example, the beautiful results of F. R. Sharpe and Virgil Snyder, "Certain types of involutorial space transformations," *Transactions of the American Mathematical Society*, vol. 20 (1919), pp. 185-202, follow immediately from the case $n = k = 3$, $h = 2$, of our congruences.

PART I. *n*-Dimensional Space.

1. Fixed base. Consider the r simultaneous equations

$$(1.1)\qquad \begin{array}{c} \bar{x}_1 f_{11} + \cdots + \bar{x}_{r+1} f_{1\,r+1} = 0, \\ \cdots\cdots \\ \bar{x}_1 f_{r1} + \cdots + \bar{x}_{r+1} f_{r\,r+1} = 0, \end{array}$$

in which the f_{ij} are non-specialized forms of order ϕ_i (any positive integers) in $x_1, \cdots, x_{n+1}$, *having constant coefficients*, and the $\bar{x}_1, \cdots, \bar{x}_{r+1}$ are parameters. These equations define an ∞^r-system of varieties V in $[n]$.

For a variety V of the system (1.1) to pass through an arbitrary fixed point P of $[n]$,

$$\begin{array}{c} \bar{x}_1 p_{11} + \cdots + \bar{x}_{r+1} p_{1\,r+1} = 0, \\ \cdots\cdots \\ \bar{x}_1 p_{r1} + \cdots + \bar{x}_{r+1} p_{r\,r+1} = 0, \end{array}$$

where p_{ij} is the result of substituting the coördinates of P in f_{ij}. Hence

$$\bar{x}_1 : \bar{x}_2 : \cdots : \bar{x}_{r+1} = P_1 : P_2 : \cdots : P_{r+1},$$

where P_j is $(-1)^{j+1}$ times the determinant formed from the matrix

$$\left\| \begin{array}{ccc} p_{11} & \cdots & p_{1\,r+1} \\ \cdot & \cdots & \cdot \\ p_{r1} & \cdots & p_{r\,r+1} \end{array} \right\|$$

by omitting the j-th column.

That is, the equations of the variety V of the ∞^r-system (1.1) passing through an arbitrary fixed point P of $[n]$ are

$$(1.2)\qquad \begin{array}{c} P_1 f_{11} + \cdots + P_{r+1} f_{1\,r+1} = 0, \\ \cdots\cdots \\ P_1 f_{r1} + \cdots + P_{r+1} f_{r\,r+1} = 0. \end{array}$$

Let Q be any other point on the variety V through P. From (1.2),

$$P_1 : P_2 : \cdots : P_{r+1} = Q_1 : Q_2 : \cdots : Q_{r+1}.$$

The ratios $P_1 : P_2 : \cdots : P_{r+1}$ are unique for the variety V passing through P.

The totality of the varieties (1.2) for all positions of P in $[n]$ form an ∞^r-system of which one and only one member passes through a generic point of $[n]$. Moreover, we have seen that any other point Q on the variety through

P determines that same variety. Therefore, *the equations* (1.2) *represent the variety through a generic point P of a variety congruence of order one in* $[n]$.

A generic variety of this congruence is of dimension $n-r$ and order $\prod_1^r \phi_i$.

The symbol

$$\left\|\begin{matrix} f_{11} & \cdot & \cdot & \cdot & f_{1\,r+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{r1} & \cdot & \cdot & \cdot & f_{r\,r+1} \end{matrix}\right\| = 0 \tag{1.3}$$

means the $r+1$ simultaneous equations, $F_j = 0$ $(j = 1, \cdots, r+1)$, where F_j *is* $(-1)^{j+1}$ times the determinant formed from the matrix of (1.3) by omitting the j-th column.

The locus (1.3) will be called a *fixed base* of the congruence.

For a point P, not on the fixed base (1.3), at least one P_j fails to vanish; call such a one P_g. If all the other P_j vanish, it is clear that the intersection of the variety (1.2) with the fixed base (1.3) is the same as the intersection of (1.2) with $F_g = 0$.

Should another P_j fail to vanish, say $P_k \neq 0$, by solving equations (1.2) for the f_{ik} $(i = 1, \cdots, r)$, and substituting these results in $F_g = 0$, we obtain $P_g F_k / P_k = 0$. Therefore, any point common to the variety (1.3) and $F_g = 0$ also lies on $F_k = 0$. It follows that the intersection of a generic variety (1.2) of the congruence with the fixed base (1.3) is the same as the intersection of (1.2) with any $F_j = 0$ for which $P_j \neq 0$. *Every variety* (1.2) *of the congruence intersects the fixed base* (1.3) *in a variety of dimension* $n-r-1$ *and order* $(\prod_1^r \phi_i)(\sum_1^r \phi_i)$, *which varies on the fixed base* (1.3) *as* (1.2) *varies in* $[n]$.

The equations (1.1) establish a (1,1) correspondence between the varieties V of our congruence in $[n]$ and the points $\bar{P}$, with coördinates $\bar{x}_1, \cdots, \bar{x}_{r+1}$, of a flat space $[\bar{r}]$.

A point P of $[\bar{r}]$ is determined by its coördinates; it can also be determined by any r independent primes that pass through it. We have already seen that the coördinates of $\bar{P}$ determine in equations (1.1) a variety V of the congruence in $[n]$. Likewise, r independent primes of the system

$$\lambda_1 \bar{x}_1 + \cdots + \lambda_{r+1} \bar{x}_{r+1} = 0 \tag{1.4}$$

in $[\bar{r}]$, have as correspondents in $[n]$ r primals of the ∞^r-system

$$\lambda_1 F_1 + \cdots + \lambda_{r+1} F_{r+1} = 0, \tag{1.5}$$

where the λ_j are parameters.

The r primals

$$(1.6)\qquad \begin{aligned} p_{11}F_1 + \cdots + p_{1\,r+1}F_{r+1} &= 0, \\ \cdots\cdots & \\ p_{r1}F_1 + \cdots + p_{r\,r+1}F_{r+1} &= 0, \end{aligned}$$

of the system (1.5) intersect in the fixed base (1.3) and the variety V of our congruence that passes through P. Thus, *a generic variety of our variety congruence of order one in* $[n]$ *can be represented in two ways*: *first*, as the complete intersection of the r primals (1.2), of orders ϕ_i $(i = 1, \cdots, r)$ respectively; *second*, as the variable intersection of the r primals (1.6) of the ∞^r-system (1.5) through the fixed base (1.3), all of the same order $\sum_1^r \phi_i$.

When $n = r$ and $\phi_1 = \phi_2 = \cdots = \phi_r = 1$, the ∞^r primals (1.5) form a homaloidal system, any r members of which intersect in a single point. This establishes a Cremona transformation between the points of $[n]$ and the points of $[\bar{r}]$.[3]

2. Other congruences. When $r < n$ in **1**, other variety congruences of order one in $[n]$ can be constructed. Let $r + k \leq n$.

The equations

$$(2.1)\qquad \begin{aligned} \bar{x}_1^{(2)}d_{11}^{(2)} + \cdots + \bar{x}_{k+1}^{(2)}d_{1\,k+1}^{(2)} &= 0, \\ \cdots\cdots & \\ \bar{x}_1^{(2)}d_{k1}^{(2)} + \cdots + \bar{x}_{k+1}^{(2)}d_{k\,k+1}^{(2)} &= 0, \end{aligned}$$

in which the $d_{ij}^{(2)}$ are forms of order $\phi_i^{(2)}$ (any positive integers) in $x_1, \cdots, x_{n+1}$, *having coefficients that are themselves forms* of order σ_{21i} in $\bar{x}_1, \cdots, \bar{x}_{r+1}$, ($\sigma_{21i}$ any positive integers or zero), define, for each set of ratios $\bar{x}_1 : \bar{x}_2 : \cdots : \bar{x}_{r+1}$, an ∞^k-system of varieties in $[n]$. The $\bar{x}_1^{(2)}, \cdots, \bar{x}_{k+1}^{(2)}$ are parameters of the system.

An arbitrary fixed point P of $[n]$ determines in (1.1) the ratios $\bar{x}_1 : \bar{x}_2 : \cdots : \bar{x}_{r+1} = P_1 : P_2 : \cdots : P_{r+1}$. Since all points on any one variety of the ∞^r-system discussed in **1** give the same set of ratios, equations (2.1) associate with each variety of the ∞^r-system of **1** a unique ∞^k-system of varieties. The ∞^k-system associated with the variety (1.2) through P is given by

$$(2.2)\qquad \begin{aligned} \bar{x}_1^{(2)}f_{11}^{(2)} + \cdots + \bar{x}_{k+1}^{(2)}f_{1\,k+1}^{(2)} &= 0, \\ \cdots\cdots & \\ \bar{x}_1^{(2)}f_{k1}^{(2)} + \cdots + \bar{x}_{k+1}^{(2)}f_{k\,k+1}^{(2)} &= 0, \end{aligned}$$

[3] T. G. Room, *The Geometry of Determinantal Loci*, Cambridge, 1938, p. 116.

where $f_{ij}^{(2)}$ is the result of substituting $P_1, \cdots, P_{r+1}$ for $\bar{x}_1, \cdots, \bar{x}_{r+1}$ in $d_{ij}^{(2)}$.

The entire discussion of **1** applies equally to the ∞^k-system (2.2). The unique variety of the system (2.2) passing through P has equations

$$(2.3) \qquad \begin{gathered} P_1^{(2)} f_{11}^{(2)} + \cdots + P_{k+1}^{(2)} f_{1\,k+1}^{(2)} = 0, \\ \cdots\cdots\cdots\cdots \\ P_1^{(2)} f_{k1}^{(2)} + \cdots + P_{k+1}^{(2)} f_{k\,k+1}^{(2)} = 0, \end{gathered}$$

where $P_j^{(2)}$ is $(-1)^{j+1}$ times the determinant formed from the matrix

$$\begin{Vmatrix} p_{11}^{(2)} & \cdots & p_{1\,k+1}^{(2)} \\ \cdot & \cdots & \cdot \\ p_{k1}^{(2)} & \cdots & p_{k\,k+1}^{(2)} \end{Vmatrix}$$

by omitting the j-th column. Here $p_{ij}^{(2)}$ is obtained by substituting the coördinates of P for $x_1, \cdots, x_{n+1}$ in $f_{ij}^{(2)}$.

Equations (1.2) *and* (2.3) *together constitute a first representation of the variety through P, of a variety congruence of order one in* $[n]$. A generic variety V of this congruence is of dimension $n - r - k$ and order $(\prod_1^r \phi_i)(\prod_1^k \phi_i^{(2)})$.

A second representation of this variety through P of the congruence is given by equations (1.6) and the equations

$$(2.4) \qquad \begin{gathered} p_{11}^{(2)} F_1^{(2)} + \cdots + p_{1\,k+1}^{(2)} F_{k+1}^{(2)} = 0, \\ \cdots\cdots\cdots\cdots \\ p_{k1}^{(2)} F_1^{(2)} + \cdots + p_{k\,k+1}^{(2)} F_{k+1}^{(2)} = 0, \end{gathered}$$

in which $F_j^{(2)}$ is $(-1)^{j+1}$ times the determinant formed from the matrix

$$\begin{Vmatrix} f_{11}^{(2)} & \cdots & f_{1\,k+1}^{(2)} \\ \cdot & \cdots & \cdot \\ f_{k1}^{(2)} & \cdots & f_{k\,k+1}^{(2)} \end{Vmatrix}$$

by omitting the j-th column.

Thus, *a generic variety of our variety congruence of order one in* $[n]$ *can be represented in two ways*: *first*, as the complete intersection of the $r + k$ primals, (1.2) of order ϕ_i $(i = 1, \cdots, r)$ respectively, and (2.3) of orders $\phi_i^{(2)}$ $(i = 1, \cdots, k)$ respectively; *second*, as the variable intersection of r primals of the system (1.5) and k primals of the system

$$(2.5) \qquad \mu_1 F_1^{(2)} + \cdots + \mu_{k+1} F_{k+1}^{(2)} = 0,$$

The symbol

$$(2.6)\qquad \left\|\begin{matrix} d_{11}{}^{(2)} & \cdot & \cdot & \cdot & d^{(2)}_{1\,k+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{k1}{}^{(2)} & \cdot & \cdot & \cdot & d^{(2)}_{k\,k+1} \end{matrix}\right\| = 0,$$

means the $k+1$ simultaneous equations $D_j{}^{(2)} = 0$ $(j = 1, \cdots, k+1)$, where $D_j{}^{(2)}$ is $(-1)^{j+1}$ times the determinant formed from the matrix of (2.6) by omitting the j-th column.

For all points P on any one variety of the congruence of **1** equations (2.6) define one base, while for all points P on any other variety of the congruence of **1** they define a different base. The effect of this is to associate with each variety of the congruence of **1** a unique base. Accordingly, (2.6) will be called a *dependent base.*

3. Dependent base. If $r + k < n$, the process of **2** may be repeated.

In general, a dependent base is defined by

$$(3.1)\qquad \left\|\begin{matrix} d_{11}{}^{(g)} & \cdot & \cdot & \cdot & d^{(g)}_{1\,s+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{s1}{}^{(g)} & \cdot & \cdot & \cdot & d^{(g)}_{s\,s+1} \end{matrix}\right\| = 0,$$

$(1 < g \leq w)$, in which the $d_{ij}{}^{(g)}$ are forms of order $\phi_i{}^{(g)}$ (any positive integers) in $x_1, \cdots, x_{n+1}$, *whose coefficients are themselves homogeneous of order* σ_{g1i} *in* $\bar{x}_j$ $(i = 1, \cdots, \alpha_1;\ j = 1, \cdots, \alpha_1 + 1)$, *homogeneous of order* σ_{g2i} *in* $\bar{x}_j{}^{(2)}$ $(i = 1, \cdots, \alpha_2;\ j = 1, \cdots, \alpha_2 + 1), \cdots$, *homogeneous of order* $\sigma_{g\,g-1\,i}$ *in* $\bar{x}_j{}^{(g-1)}$ $(i = 1, \cdots, \alpha_{g-1};\ j = 1, \cdots, \alpha_{g-1} + 1)$.

The $\bar{x}_j$ are the parameters of the system (1.1). The $\bar{x}_j{}^{(a)}$ $(j = 1, \cdots, \alpha_a + 1)$ are parameters of the ∞^t-system $(t = \alpha_a)$

$$(3.2)\qquad \begin{matrix} \bar{x}_1{}^{(a)} d_{11}{}^{(a)} + \cdots + \bar{x}^{(a)}_{t+1} d^{(a)}_{1\,t+1} = 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \bar{x}_1{}^{(a)} d_{t1}{}^{(a)} + \cdots + \bar{x}^{(a)}_{t+1} d^{(a)}_{t\,t+1} = 0, \end{matrix}$$

for $a = 2, \cdots, g-1$. The $d_{ij}{}^{(a)}$ are forms of order $\phi_i{}^{(a)}$ (any positive integers) in $x_1, \cdots, x_{n+1}$, whose coefficients are themselves homogeneous of order σ_{a1i} in $\bar{x}_j$ $(i = 1, \cdots, \alpha_1;\ j = 1, \cdots, \alpha_1 + 1)$, homogeneous of order σ_{a2i} in $\bar{x}_j{}^{(2)}$ $(i = 1, \cdots, \alpha_2;\ j = 1, \cdots, \alpha_2 + 1), \cdots$, homogeneous of order $\sigma_{a\,a-1\,i}$ in $\bar{x}_j{}^{(a-1)}$ $(i = 1, \cdots, \alpha_{a-1};\ j = 1, \cdots, \alpha_{a-1} + 1)$.

In constructing variety congruences of order one in $[n]$ we commence with a fixed base (1.3) and continue with dependent bases (3.1) whose

matrices have $\alpha_2, \alpha_3, \cdots, \alpha_w$ rows, respectively, subject to the restriction that $\alpha_1 + \cdots + \alpha_w \leq n$.

We defined P_j in **1** and $P_j^{(2)}$ in **2**. $P_j^{(3)}$ is obtained by substituting the coördinates of P for $x_1, \cdots, x_{n+1}$ in $F_j^{(3)}$, and $F_j^{(3)}$ is $(-1)^{j+1}$ times the determinant formed from the matrix

$$\begin{Vmatrix} f_{11}^{(3)} & \cdot & \cdot & \cdot & f^{(3)}_{1\ t+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{t1}^{(3)} & \cdot & \cdot & \cdot & f^{(3)}_{t\ t+1} \end{Vmatrix}$$

by omitting the j-th column. Here $f_{ij}^{(3)}$ is obtained from $d_{ij}^{(3)}$ by substituting P_k for $\bar{x}_k$ and $P_k^{(2)}$ for $\bar{x}_k^{(2)}$. Similarly $P_j^{(4)}, P_j^{(5)}, \cdots$, are defined.

The unique variety V of our variety congruence of order one in $[n]$ that passes through an arbitrary fixed point P is given by equations (1.2) together with

$$\begin{aligned} &P_1^{(g)} f_{11}^{(g)} + \cdots + P^{(g)}_{s+1} f^{(g)}_{1\ s+1} = 0, \\ (3.3)\qquad &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &P_1^{(g)} f_{s1}^{(g)} + \cdots + P^{(g)}_{s+1} f^{(g)}_{s\ s+1} = 0, \end{aligned}$$

for $g = 2, \cdots, w$. $f_{ik}^{(g)}$ is obtained from $d_{ik}^{(g)}$ by substituting P_j for $\bar{x}_j$, and $P_j^{(a)}$ for $x_j^{(a)}$ $(a = 2, \cdots, g-1)$.

A generic variety of this congruence is of dimension $n - (\alpha_1 + \cdots + \alpha_w)$. In general, its order is the product of all the $\phi_i^{(k)}$.

If all the elements of any one column of (3.1) contain the same form in $x_j^{(k)}$ $(j = 1, \cdots, \alpha_k + 1)$ as factor, this factor will fall away from all the equations (3.3). By removing such factors from the columns of (3.1) we obtain what was recently caled a terminal base.[4]

4. Classification. The type symbol $(\alpha_1 \cdots \alpha_w)_n$ denotes a variety congruence of order one in $[n]$; α_1 is the number of rows in the matrix of the fixed base (1.3), α_g $(g = 2, \cdots, w)$ is the number of rows in the matrix of the dependent base (3.1). The α_i are positive integers whose sum k is not greater than n.

In what has gone before, the congruences have consisted of varieties of dimension $n - k$, whose first representation is the *complete* intersection of k primals. However, this may be modified so that the first representation of a generic variety is a *partial* intersection of k primals, as will be shown by an example in **7**.

When the generic variety V of the type $(\alpha_1 \cdots \alpha_w)_n$ congruence of order

[4] Purcell, *loc. cit.*, § 4.

one in $[n]$ is the complete intersection of $k = \sum_1^w \alpha_i$ primals, it is of dimension $n - k$ and order

$$\nu = (\prod_1^{\alpha_1} \phi_i)(\prod_1^{\alpha_2} \phi_i^{(2)}) \cdots (\prod_1^{\alpha_w} \phi_i^{(w)}).$$

In general, V intersects the bases in a variety consisting of w parts, each of dimension $n - k - 1$, whose orders are, respectively,

$$\nu(\sum_1^{\alpha_1} \phi_i),\ \nu(\sum_1^{\alpha_2} \phi_i^{(2)}), \cdots, \nu(\sum_1^{\alpha_w} \phi_i^{(w)}).$$

But, under certain circumstances, some or all of these w parts may be required to coincide.[5]

For type $(\alpha_1 \cdots \alpha_w)_n$, the ϕ_i and $\phi_i^{(g)}$ $(g = 2, \cdots, w)$ are any positive integers, and the σ_{gki} are any positive integers or zero.

5. Fundamental loci. A point through which more than one variety of the congruence passes is called a *fundamental point*. A fundamental point fails to determine a unique variety of the congruence.

Two kinds of fundamental points in $[n]$ will be considered. Those of the *first kind* fail to determine some or all of the equations (1.2) and (3.3). Those of the *second kind* determine all the equations (1.2) and (3.3) but the equations are not independent.

As the arbitrary point P varies in position throughout $[n]$, the intersection of the dependent base (3.1) with its associated variety

$$(5.1)\qquad \begin{aligned} &P_1^{(a)} f_{11}^{(a)} + \cdots + P_{d+1}^{(a)} f_{1\,d+1}^{(a)} = 0,\\ &\cdots\cdots\cdots\cdots\\ &P_1^{(a)} f_{d1}^{(a)} + \cdots + P_{d+1}^{(a)} f_{d\,d+1}^{(a)} = 0,\end{aligned}$$

$(a = 1, \cdots, g - 1)$, sweeps out a locus which we will call a *dependent base locus* and designate by B_g $(1 < g \leq w)$.

B_1 denotes the fixed base (1.3).

The points of B_1 and B_g $(g = 2, \cdots, w)$ are fundamental points of the first kind for the variety congruence of order one in $[n]$. These fundamental varieties are of dimension $n - 2$. In special cases, however, the dimension of a fundamental variety may be lower, as will be shown in **7**.

The symbol

$$(5.2)\qquad \left\| \begin{matrix} h_{11}^{(g)} & \cdots & h_{1\,s+1}^{(g)} \\ \cdots & \cdots & \cdots \\ h_{s1}^{(g)} & \cdots & h_{s\,s+1}^{(g)} \end{matrix} \right\| = 0$$

[5] Purcell, *loc. cit.*, § 6.

means the $s+1$ simultaneous equations $H_j^{(g)}=0$ $(j=1,\cdots,s+1)$, where $H_j^{(g)}$ is $(-1)^{j+1}$ times the determinant formed from the matrix of (5.2) by omitting the j-th column. The $h_{ij}^{(g)}$ are obtained by changing the coördinates of P to $x_1,\cdots,x_{n+1}$ wherever they appear in $f_{ij}^{(g)}$.

The equations (5.2) *give the dependent base locus* B_g $(g=2,\cdots,w)$, *after all preceding dependent base loci* B_{g-i} *have been rejected.*

To find the order of B_1, the fixed base (1.3), choose any two $F_j=0$. As noted by Salmon [6] in a somewhat similar situation, B_1 is the residual intersection of these primals after rejecting the locus expressed by the simultaneous vanishing of all the $(r-1)$-row determinants common to the two selected F_j. It follows by induction that *the order of* B_1, *the fixed base* (1.3), *is equal to the sum of the squares of the* ϕ_i *plus the sum of the products of the* ϕ_i *taken two at a time.*

Formulae for the orders of the dependent base loci, B_g $(g=2,\cdots,w)$, can be found but are cumbersome. Instead, a method for their computation will be indicated by an example.

Consider the type $(1\,2)_4$ conic congruence of order one in [4], in which $\phi_1=2$, $\phi_1^{(2)}=\phi_2^{(2)}=1$. The fixed base B_1 is the quartic surface

$$\| f_{11} \quad f_{12} \| = 0, \tag{5.3}$$

basis for a pencil of hyperquadrics.

The hyperquadric of this pencil passing through an arbitrary point P of [4] is

$$P_1 f_{11} + P_2 f_{12} = 0. \tag{5.4}$$

The dependent base is

$$\left\| \begin{matrix} d_{11}^{(2)} & d_{12}^{(2)} & d_{13}^{(2)} \\ d_{21}^{(2)} & d_{22}^{(2)} & d_{23}^{(2)} \end{matrix} \right\| = 0. \tag{5.5}$$

For each hyperquadric (5.4), equations (5.5) represent a rational normal cubic surface in [4], and the plane through P, intersecting it in a conic, is given by

$$\begin{aligned} P_1^{(2)} f_{11}^{(2)} + P_2^{(2)} f_{12}^{(2)} + P_3^{(2)} f_{13}^{(2)} &= 0, \\ P_1^{(2)} f_{21}^{(2)} + P_2^{(2)} f_{22}^{(2)} + P_3^{(2)} f_{23}^{(2)} &= 0. \end{aligned} \tag{5.6}$$

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[6] G. Salmon, *Modern Higher Algebra*, 4th ed., Dublin, 1885, § 272.

The equations (5.4) and (5.6) simultaneously define the unique conic of the congruence through an arbitrary point P of [4].

The dependent base locus, B_2, is of dimension 2. Its equations are

$$(5.7)\qquad \left\| \begin{matrix} h_{11}^{(2)} & h_{12}^{(2)} & h_{13}^{(2)} \\ h_{21}^{(2)} & h_{22}^{(2)} & h_{23}^{(2)} \end{matrix} \right\| = 0.$$

In finding the order of B_2, (5.7), notice that $H_1^{(2)} = 0$ is a primal of order $2(\sigma_{211} + \sigma_{212} + 1)$ on which the quartic surface B_1, (5.3), is $(\sigma_{211} + \sigma_{212})$-fold. $H_2^{(2)} = 0$ is a primal of order $2(\sigma_{211} + \sigma_{212} + 1)$ containing (5.3) as $(\sigma_{211} + \sigma_{212})$-fold surface. $H_1^{(2)} = 0$ and $H_2^{(2)} =$ intersect in surfaces of total order $4(\sigma_{211} + \sigma_{212} + 1)^2$, containing (5.3) counted $(\sigma_{211} + \sigma_{212})^2$ times, from which must be rejected the surface whose equations are $h_{13}^{(2)} = h_{23}^{(2)} = 0$. This rejected surface is of order $(2\sigma_{211} + 1) \times (2\sigma_{212} + 1)$, containing (5.3) counted $\sigma_{211}\sigma_{212}$ times. The residual intersection of $H_1^{(2)} = 0$ and $H_2^{(2)} = 0$ is of order $4\sigma_{211}^2 + 4\sigma_{212}^2 + 4\sigma_{211}\sigma_{212} + 6\sigma_{211} + 6\sigma_{212} + 3$, containing (5.3) as $(\sigma_{211}^2 + \sigma_{211}\sigma_{212} + \sigma_{212}^2)$-fold surface. Therefore, *the dependent base locus, B_2, is a surface of order* $6\sigma_{211} + 6\sigma_{212} + 3$.

The dependent base locus, B_2, intersects the fixed base, B_1, in a curve of order $12(\sigma_{211} + \sigma_{212})$.

B_1 and B_2 are surfaces of fundamental points of the first kind. When they are given, the congruence is completely determined.

Fundamental points of the second kind are those points of $[n]$, not on $B_1, \cdots, B_w$, for which the equations (1.2) and (3.3) fail to be independent. In such cases, the variety defined by some of these equations is contained completely in the variety defined by the remaining equations. In the above example of a type $(1\ 2)_4$ conic congruence of order one in [4], let $\sigma_{211} = \sigma_{212} = 0$. Then B_2 is a rational normal cubic surface in [4]. There are 20 planes of the system (5.6) that lie entirely on their associated hyperquadric (5.4). Each such plane intersects B_1 in a conic and B_2 in a conic. Through any point of such a plane, not on B_1 or B_2, ∞^4 conics may be drawn intersecting B_1 in 4 points and B_2 in 4 points. Thus no point on any of these 20 planes determines a unique conic of the congruence. The points of these 20 planes, not on B_1 or B_2, are fundamental points of the second kind for the conic congruence.[7]

[7] Another example of fundamental points of the second kind is given in Purcell, *loc. cit.*, § 2.

6. An application. As has already been indicated, when $n = k$ and $h = 2$ the generic variety of any of our congruences is a $V_0{}^2$. Each of the two points determines the pair, and P and P' are correspondents in a Cremona involution in $[n]$. An extensive treatment of Cremona involutions from this point of view is now in preparation, but it may be worthwhile to give one example here to illustrate the usefulness of our theory in new discovery.

Our type $(1\,1\,1)_3$ variety congruence of order one, in which $\phi_1 = 1$, $\phi_1{}^{(2)} = 2$, and $\phi_1{}^{(3)} = 1$, gives a very general Cremona involution in $[3]$. This involution does not seem to be in the literature, although Sharpe and Snyder's III_1 involution is a specialization of it.[8]

The fixed base is

$$B_1 \equiv \| f_{11} \quad f_{12} \| = 0, \tag{6.1}$$

where $f_{1j} \equiv \sum_1^4 a_{1jk}x_k$, and a_{1jk} are constants $(k = 1, \cdot\cdot, 4)$.

The ∞^1-system based on (6.1) is

$$\bar{x}_1 f_{11} + \bar{x}_2 f_{12} = 0. \tag{6.2}$$

The first dependent base is

$$\| d_{11}{}^{(2)} \quad d_{12}{}^{(2)} \| = 0, \tag{6.3}$$

where $d_{1j}{}^{(2)} \equiv \sum_{k,l}^{1-4} b_{jkl}x_k x_l$, in which the b_{jkl} are homogeneous of order σ_{211} in $\bar{x}_1, \bar{x}_2$ (σ_{211} any positive integer or zero).

The ∞^1-system based on (6.3) is

$$\bar{x}_1{}^{(2)} d_{11}{}^{(2)} + \bar{x}_2{}^{(2)} d_{12}{}^{(2)} = 0. \tag{6.4}$$

The second dependent base is

$$\| d_{11}{}^{(3)} \quad d_{12}{}^{(3)} \| = 0, \tag{6.5}$$

where $d_{1j}{}^{(3)} \equiv \sum_1^4 a_{1jk}^{(3)} x_k$, and $a_{1jk}^{(3)}$ is homogeneous of order σ_{311} in $\bar{x}_1, \bar{x}_2$, and also homogeneous of order σ_{321}, in $\bar{x}_1{}^{(2)}, \bar{x}_2{}^{(2)}$, $(k = 1, \cdot\cdot, 4)$. The σ_{321} and σ_{311} are any positive integers or zero.

The ∞^1-system based on (6.5) is

$$\bar{x}_1{}^{(3)} d_{11}{}^{(3)} + \bar{x}_2{}^{(3)} d_{12}{}^{(3)} = 0. \tag{6.6}$$

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[8] Sharpe and Snyder, *loc. cit.*, p. 201.

Equations (6.2) and (6.6) can be rewritten

$$\bar{m}_{11}x_1 + \bar{m}_{12}x_2 + \bar{m}_{13}x_3 + \bar{m}_{14}x_4 = 0, \tag{6.7}$$

$$\bar{m}_{21}x_1 + \bar{m}_{22}x_2 + \bar{m}_{23}x_3 + \bar{m}_{24}x_4 = 0, \tag{6.8}$$

respectively, in which $\bar{m}_{1k} \equiv a_{11k}\bar{x}_1 + a_{12k}\bar{x}_2$, and

$$\bar{m}_{2k} \equiv a^{(3)}_{11k}\bar{x}_1{}^{(3)} + a^{(3)}_{12k}\bar{x}_2{}^{(3)} \qquad (k = 1, \cdot \cdot, 4).$$

Denote by $\bar{M}$ the matrix

$$\bar{M} \equiv \left\| \begin{matrix} \bar{m}_{11} & \bar{m}_{12} & \bar{m}_{13} & \bar{m}_{14} \\ \bar{m}_{21} & \bar{m}_{22} & \bar{m}_{23} & \bar{m}_{24} \end{matrix} \right\|.$$

We define $\bar{M}_{\alpha\beta}$, for $\alpha < \beta$, to be $(-1)^{\alpha+\beta}$ times the determinant formed from the matrix $\bar{M}$ by omitting the α-th and β-th columns; $\bar{M}_{\alpha\beta}$ for $\alpha = \beta$ is defined as zero; $\bar{M}_{\alpha\beta}$ for $\alpha > \beta$ is defined to be $-\bar{M}_{\beta\alpha}$.

Equation (6.4) can be rewritten

$$\sum_{i,j}^{1-4} \bar{A}_{ij}x_ix_j = 0, \tag{6.9}$$

where $\bar{A}_{kl} \equiv b_{1kl}\bar{x}_1{}^{(2)} + b_{2kl}\bar{x}_2{}^{(2)}$.

From equations (6.7), (6.8), and (6.9) we obtain *the equations of the involution*,

$$\rho x'_r = (\sum_{i,j}^{1-4} A_{ij}M_{ir}M_{jr}) \div x_r, \tag{6.10}$$

$(r = 1, \cdot \cdot \cdot, 4)$, in which A_{ij}, M_{ir}, and M_{jr} are obtained from $\bar{A}_{ij}$, $\bar{M}_{ir}$, and $\bar{M}_{jr}$, respectively, by substituting F_j for $\bar{x}_j$, $H_j{}^{(2)}$ for $\bar{x}_j{}^{(2)}$, and $H_j{}^{(3)}$ for $\bar{x}_j{}^{(3)}$, $(j = 1, 2)$.

The order of the involution is $4\sigma_{321}\sigma_{211} + 8\sigma_{321} + 4\sigma_{311} + 2\sigma_{211} + 5$.

The fixed base is a line of fundamental points of the first kind. B_1 is a $(4\sigma_{321}\sigma_{211} + 4\sigma_{311} + 2\sigma_{211} + 2)$-fold line on every homaloid.

The dependent base locus, B_2, is a curve of fundamental points of the first kind. B_2 is a curve of order $4\sigma_{211} + 4$, having $4\sigma_{211}$ points on line B_1. B_2 is $(4\sigma_{321} + 1)$-fold on every homaloid.

The second dependent base locus, B_3, is a curve of fundamental points of the first kind. B_3 is a curve of order $4\sigma_{321}\sigma_{311} + 2\sigma_{321}\sigma_{211} + 4\sigma_{321} + 2\sigma_{311} + 1$, having $4\sigma_{321}\sigma_{311} + 2\sigma_{321}\sigma_{211} + 2\sigma_{311}$ points on line B_1 and $8\sigma_{321}(\sigma_{311} + \sigma_{211} + 1)$ points on curve B_2. B_3 is a double curve on every homaloid.

Fundamental points of the second kind are those points, not on B_1, B_2, or B_3, for which equations (6.7), (6.8), and (6.9) fail to be independent. Any point of [3], not on B_1, B_2, or B_3, determines, by means of equations

(6. 7) and (6. 8), a line passing through it. This line, in general, intersects its associated surface (6. 9) in two free points, P and P'. But a finite number of such lines lie entirely on their associated quadrics. To any point of such a line there corresponds the whole line. All points on such lines, not on B_1, B_2, or B_3, are fundamental points of the second kind.

PART II. Curve Congruences of Order One in [3].

There are two types of curve congruences of order one in [3], as indicated by the type symbols $(2)_3$ and $(11)_3$.

7. Type $(2)_3$. The fixed base is

$$\left\| \begin{matrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{matrix} \right\| = 0, \tag{7.1}$$

and the unique curve of the congruence passing through an arbitrary point P of [3] is given by

$$\begin{aligned} P_1f_{11} + P_2f_{12} + P_3f_{13} = 0, \\ P_1f_{21} + P_2f_{22} + P_3f_{23} = 0. \end{aligned} \tag{7.2}$$

Two cases of type $(2)_3$ must be considered according as all the elements of either row of the matrix of (7. 1) do not, or do, have a fixed curve in common.

In the first case there is no fixed curve common to all the $f_{1j} = 0$ or common to all the $f_{2j} = 0$. A generic curve (7. 2) of the congruence is of order $\phi_1\phi_2$ and the fixed base (7. 1) is a space curve of order $\phi_1^2 + \phi_1\phi_2 + \phi_2^2$. The fixed base (7. 1) is a curve of fundamental points of the first kind.

The $f_{1j} = 0$ intersect in one set of ϕ_1^3 associated points and the $f_{2j} = 0$ intersect in a second set of ϕ_2^3 associated points, $(j = 1, 2, 3)$. When no point of the first set coincides with a point of the second set, the fixed base (7. 1) passes simply through all points of both sets. A generic curve of the congruence intersects the fixed base (7. 1) in $\phi_1\phi_2(\phi_1 + \phi_2)$ variable points. If a point K of the first set of associated points coincides with a point of the second set, the fixed base (7. 1) has K as a triple point and every curve of the congruence goes through this fixed point. The number of variable intersections of a generic curve of the congruence with the fixed base curve is reduced by 3. This process continues for each point of the first set that coincides with a point of the second set.

When $\phi_1 = \phi_2$, an interesting special case arises if the two sets of ϕ_1^3 associated points are the same—that is, if all the $f_{ij} = 0$ belong to the same

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net. The order of the curve C whose equations are (7.1) cannot be greater than $3\phi_1^2$, and C has the ϕ_1^3 associated points as triple points. Consider a pencil of surfaces of order ϕ_1 through the ϕ_1^3 associated points and another point on C. Each surface of this pencil intersects C in $3\phi_1^3 + 1$ points and therefore C lies entirely on every surface of the pencil. The order of C is ϕ_1^2. C is not a fundamental curve but merely a member of the curve congruence. The ϕ_1^3 associated points are the only fundamental points and these are isolated. This curve congruence of order one in [3], consisting of the curves of order n^2 through n^3 associated points, has long been known and is now seen to be a special case of our type $(2)_3$.

In the second case of type $(2)_3$, one or more curves lie on all the $f_{1j} = 0$ or on all the $f_{2j} = 0$. Let Γ be a fixed curve of order γ common to all the $f_{1j} = 0$. The fixed base (7.1) is now composite, consisting of the curve Γ counted once and a residual curve of order $\phi_1^2 + \phi_1\phi_2 + \phi_2^2 - \gamma$.

Should Γ lie on all the $f_{1j} = 0$ and *also* on all the $f_{2j} = 0$, the fixed base (7.1) will consist of the curve Γ counted three times and a residual curve of order $\phi_1^2 + \phi_1\phi_2 + \phi_2^2 - 3\gamma$. In this case a generic curve of the congruence is the partial intersection of the two surfaces (7.2), the residual in each instance being the fixed curve Γ counted once.

8. Type $(1\,1)_3$. The fixed base for the type $(1\,1)_3$ congruence is

$$(8.1) \qquad \| f_{11} \quad f_{12} \| = 0$$

and the equation

$$(8.2) \qquad P_1 f_{11} + P_2 f_{12} = 0$$

represents a surface of order ϕ_1, through an arbitrary point P of [3], belonging to the pencil on (8.1).

The dependent base is

$$(8.3) \qquad \| d_{11}^{(2)} \quad d_{12}^{(2)} \| = 0.$$

For each surface (8.2) of the pencil on (8.1), the equations (8.3) define a unique curve of order $(\phi_1^{(2)})^2$, basis of a pencil of surfaces. The surface of this pencil through P is

$$(8.4) \qquad P_1^{(2)} f_{11}^{(2)} + P_2^{(2)} f_{12}^{(2)} = 0,$$

which is of order $\phi_1^{(2)}$.

Equations (8.2) and (8.4) simultaneously define the unique curve of the congruence passing through an arbitrary point P of [3].

The locus of fundamental points of the congruence is the fixed base, B_1, whose equations are (8.1), and the dependent base locus, B_2, whose equations are

$$(8.5) \qquad \| h_{11}^{(2)} \quad h_{12}^{(2)} \| = 0.$$

The fixed base, B_1, is a curve of order ϕ_1^2 and the dependent base locus, B_2, is a curve of order $(\phi_1^{(2)})^2 + 2\phi_1^{(2)}\sigma_{211}\phi_1$. B_1 and B_2 intersect in $2\phi_1^{(2)}\sigma_{211}\phi_1^2$ points.

Should $d_{11}^{(2)}$ have as factor a binary form of order ω $(\omega \leq \sigma_{211})$ in $\bar{x}_1, \bar{x}_2$, the dependent base locus, B_2, would be a curve of order $(\phi_1^{(2)})^2 + 2\phi_1^{(2)}\sigma_{211}\phi_1 - \omega\phi_1\phi_1^{(2)}$ intersecting the fixed base, B_1, in $2\phi_1^{(2)}\sigma_{211}\phi_1^2 - \omega\phi_1^{(2)}\phi_1^2$ points.

A generic curve of the congruence is of order $\phi_1\phi_1^{(2)}$. It intersects the fixed base, B_1, in $\phi_1^{(2)}\phi_1^2$ points and the dependent base locus, B_2, in $\phi_1(\phi_1^{(2)})^2$ points. These points, in general, vary on B_1 and B_2 with the generic curve of the congruence.

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RELATIONS BETWEEN THE COMPOSITES OF A FIELD AND THOSE OF A SUBFIELD.*

By N. Jacobson.

The present note is an addendum to a recent paper appearing in this *Journal* on a Galois theory for arbitrary fields.[1] We recall that the fundamental concept of the general theory is that of a composite of a field P with itself defined to be a system $\Gamma = (K, S, T)$ consisting of a ring K and two isomorphisms S and T of P into subfields P^S and P^T of K such that 1) K is commutative, 2) $K = P^S P^T$, 3) $1^S = 1^T$, and 4) $(K : P^T)$ is finite. Now let Σ be a subfield of finite index (i. e., $(P : \Sigma)$ finite) in P; then it is readily seen that Γ determines a composite $\Gamma(\Sigma) = (\Sigma^S \Sigma^T, S, T)$ of Σ with itself. In this paper we shall investigate relations between Γ and $\Gamma(\Sigma)$. In the special case where $P \geqq \Sigma \geqq \Phi$ and P is finite and separable over Φ, the correspondence between Γ and $\Gamma(\Sigma)$ induces a homomorphism between the hypergroup $H_{P|\Phi}$ of P over Φ[2] and the hypergroup $H_{\Sigma|\Phi}$ of Σ over Φ. We show also that the hypergroup $H_{\Sigma|\Phi}$ is isomorphic to the hypergroup $H_{P|\Phi} /\!/ H_{P|\Sigma}$ of double cosets of $H_{P|\Sigma}$ in $H_{P|\Phi}$. This implies that the hypergroup of a separable field is isomorphic to the hypergroup of double cosets of a group and hence is completely regular in the sense of Dresher and Ore.[3] In the last section of this paper we give an independent proof of this fact by deriving certain properties of inverses of self-representation that may be of intrinsic interest.

1. Composites and self-representations induced in a subfield. Let P be an arbitrary field and let $\Gamma = (K, S, T)$ be a composite of P with itself. Suppose that Σ is a subfield of finite index in P. Then if $(K : P^T) = m$ and $(P : \Sigma) = q$, $(K : \Sigma^T) = mq$ and so $(\Sigma^S \Sigma^T : \Sigma^T) \leqq mq$. It follows that

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[1] "An extension of Galois theory to non-normal and non-separable fields," vol. 66 (1944), pp. 1-29, referred to as E.

[2] In a slightly different form this hypergroup was first defined by Kaloujnine in "Sur la théorie de Galois des corps nongaloisiens séparables," *Comptes Rendus de l'Académie des Sciences*, vol. 214 (1942), pp. 597-599. Cf. also E. pp. 24-26.

[3] "Theory of multigroups," this *Journal*, vol. 60 (1938), pp. 705-733.

$(\boldsymbol{\Sigma}^S\boldsymbol{\Sigma}^T, S, T)$ is a composite of $\boldsymbol{\Sigma}$ with itself. We shall denote this composite as $\boldsymbol{\Gamma}(\boldsymbol{\Sigma})$ and shall call it the *contraction* of $\boldsymbol{\Gamma}$ to the subfield $\boldsymbol{\Sigma}$.

Suppose next that E is a self-representation of $\mathbf{P}$. Then if $\gamma \in \boldsymbol{\Sigma}$, the correspondence $\gamma \rightarrow \gamma^E$ is a representation of $\boldsymbol{\Sigma}$ by matrices with elements in $\mathbf{P}$. Let R be a regular representation of $\mathbf{P}$ over $\boldsymbol{\Sigma}$. Then the elements αR_{pq} of the representing matrices $\alpha^R = (\alpha R_{pq})$ are in $\boldsymbol{\Sigma}$. Now we may replace the elements γE_{ij} of $\gamma^E = (\gamma E_{ij})$ by the matrices $\gamma E_{ij}{}^R$ and obtain in this way a self-representation $G = E \times R$ of $\boldsymbol{\Sigma}$. Let $\mathfrak{R}$ be the double $\mathbf{P}$-module and $x_1, \cdots, x_n$ the right basis of $\mathfrak{R}$ that gives rise to E. If $\rho_1, \cdots, \rho_q$ is a basis for $\mathbf{P}$ over $\boldsymbol{\Sigma}$ that gives rise to the regular representation R, then the vectors $x_1\rho_1, \cdots, x_1\rho_q$; $x_2\rho_1, \cdots, x_2\rho_q; \cdots; \cdots, x_n\rho_q$ form a right $\boldsymbol{\Sigma}$-basis for $\mathfrak{R}$. Then $\mathfrak{R}$ may be regarded as a double $\boldsymbol{\Sigma}$-module $\mathfrak{R}(\boldsymbol{\Sigma})$ and it is readily seen that the self-representation obtained from the basis $x_i\rho_j$ in the order given is $G = E \times R$. If $\boldsymbol{\Gamma} = (\mathbf{P}^E\mathbf{P}^D, E, D)$ is the composite of E, we know that $\boldsymbol{\Gamma} = (\mathbf{P}_l\mathbf{P}_r, L, R)$.[4] Since the composite of $\mathfrak{R}(\boldsymbol{\Sigma})$ is $(\boldsymbol{\Sigma}_l\boldsymbol{\Sigma}_r, L, R)$, it is clear that the composite of the self-representation G of $\boldsymbol{\Sigma}$ is the contraction of the composite of E.

Now let $\boldsymbol{\Gamma}'$ be a given composite of $\boldsymbol{\Sigma}$ with itself and let G be a self-representation of $\boldsymbol{\Sigma}$ having $\boldsymbol{\Gamma}'$ as its composite. Again let R be a regular representation of $\mathbf{P}$ over $\boldsymbol{\Sigma}$. For the element αR_{pq} of the matrix α^R we now substitute $\alpha R_{pq}{}^G$ and we obtain in this way a new self-representation E of $\mathbf{P}$ such that the elements αE_{ij} of the matrices α^E all lie in $\boldsymbol{\Sigma}$. For γ in $\boldsymbol{\Sigma}$ we have

$$\gamma^E = \left.\begin{bmatrix} \gamma^G & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \gamma^G \end{bmatrix}\right\} q \quad = \quad 1_q \times \gamma^G.$$

Hence in the self-representation $E \times R$ of $\boldsymbol{\Sigma}$ we have

$$\gamma^{E \times R} = \begin{bmatrix} \gamma^G \times 1_q & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \gamma^G \times 1_q \end{bmatrix}$$

and this matrix is similar to $1_{q^2} \times \gamma^G$. Thus the self-representation $E \times R$ of $\boldsymbol{\Sigma}$ is similar to a multiple of the given self-representation G. Hence the composite $\boldsymbol{\Gamma}'$ of G is the contraction $\boldsymbol{\Gamma}(\boldsymbol{\Sigma})$ of the composite $\boldsymbol{\Gamma}$ of E.

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[4] As in E equivalent composites are identified. The symbol $=$ is used for equivalence and we write $\Gamma_1 \geq \Gamma_2$ for "Γ_1 is a cover of Γ_2."

We recall that a composite $\Gamma = (K, S, T)$ of P is simple if $K = \mathrm{P}^S\mathrm{P}^T$ is a field. Then if Σ is a subfield of finite index in P, K is finite over Σ^T and hence $\Sigma^S\Sigma^T$ is a field. Thus the contraction $\Gamma(\Sigma)$ is simple. Conversely let Γ' be a given simple composite of Σ with itself and let Γ be a composite of P with itself such that $\Gamma(\Sigma) = \Gamma'$. Suppose that $\mathfrak{R}$ is a double P-module having the composite Γ and let $\mathfrak{S}$ be an irreducible submodule of $\mathfrak{R}$. Then if $\bar{\Gamma}$ is the composite of $\mathfrak{S}$, $\bar{\Gamma}(\Sigma) \leq \Gamma(\Sigma) = \Gamma'$. Since Γ' is simple, $\bar{\Gamma}(\Sigma) = \Gamma'$. We summarize these results in the following

THEOREM 1. *If Σ is a subfield of finite index in P and Γ' is a composite of Σ with itself, then there exists a composite Γ of P with itself whose contraction $\Gamma(\Sigma) = \Gamma'$. Moreover, if Γ' is simple, Γ may be taken to be simple.*

2. Conditions for equivalence. We recall that the composite of a self-representation E is determined by the set of endomorphisms $\Sigma E_{ij}\bar{\rho}_{ij}$ where $\bar{\rho}$ denotes the multiplication $\xi \to \xi\rho$ in P. Since $\bar{\rho}E_{ij} = \Sigma E_{i\alpha}\bar{\rho}_{\alpha j}$, it follows that a necessary and sufficient condition that $\alpha \to \alpha^E$ and $\alpha \to \alpha^F = (\alpha F_{kl})$ have equivalent composites is that each F_{kl} be expressible in the form $\Sigma E_{ij}\bar{\rho}_{ij,kl}$ and each E_{ij} have the form $\Sigma F_{kl}\bar{\tau}_{kl,ij}$. We recall also that if R is the regular self-representation of P over Σ, its composite Δ is closed and the set $\Sigma R_{pq}\bar{\rho}_{pq}$ is the complete ring of linear transformations of P over Σ. If $\gamma \in \Sigma$, $\gamma R_{pq} = \delta_{pq}\gamma$ and hence the totality $\Sigma R_{pq}\bar{\sigma}_{pq}$, σ in Σ, reduces to the set of multiplications $\bar{\sigma}$ if these endomorphisms are restricted to act in Σ. We may now prove the following

THEOREM 2. *Let Γ_1 and Γ_2 be two composites of P, Σ a subfield of finite index in P and Δ the closed composite of P over Σ. Then if*

$$\Gamma_1(\Sigma) = \Gamma_2(\Sigma), \qquad \Delta \times \Gamma_1 \times \Delta = \Delta \times \Gamma_2 \times \Delta.$$

Let E_1 and E_2 be self-representations of P with composites Γ_1 and Γ_2 respectively and let R be a regular representation of P over Σ. Then the composite of R is Δ. Suppose now that $\Gamma_1(\Sigma) = \Gamma_2(\Sigma)$. Then the induced representations $G_1 = E_1 \times R$ and $G_2 = E_2 \times R$ of Σ have the same com- composite. Hence if $E_{ij}^{(1)}, E_{kl}^{(2)}, R_{pq}$ denote the endomorphisms associated with E_1, E_2 and R respectively, there exist elements μ, ν in Σ such that

$$\gamma E_{ij}^{(1)} R_{pq} = \gamma(\Sigma E_{kl}^{(2)} R_{p'q'}\bar{\mu}_{klp'q',ijpq})$$
$$\gamma E_{kl}^{(2)} R_{pq} = \gamma(\Sigma E_{ij}^{(1)} R_{p'q'}\bar{\nu}_{ijp'q'.klpq})$$

for all γ in Σ. Since for any α in P, αR_{rs} is in Σ, it follows that these equa-

tions are valid when γ is replaced by αR_{rs}. The resulting equations show that $R \times E_1 \times R$ and $R \times E_2 \times R$ have the same composite. Thus

$$\Delta \times \Gamma_1 \times \Delta = \Delta \times \Gamma_2 \times \Delta.$$

3. Combinatorial properties of composites. It follows directly from the definitions that if Γ_1 and Γ_2 are composites of P and Γ_1 is a cover of $\Gamma_2 (\Gamma_1 \geq \Gamma_2)$ then the contraction $\Gamma_1(\Sigma) \geq \Gamma_2(\Sigma)$. If $\Gamma_1 + \Gamma_2$ denotes the least common cover of Γ_1 and Γ_2 then

$$(\Gamma_1 + \Gamma_2)(\Sigma) = \Gamma_1(\Sigma) + \Gamma_2(\Sigma).$$

Concerning multiplication of composites we have the following

THEOREM 3. *Let* Γ_1, Γ_2, Σ *and* Δ *be as in Theorem* 2. *Then*

$$(\Gamma_1 \times \Delta \times \Gamma_2)(\Sigma) = \Gamma_1(\Sigma) \times \Gamma_2(\Sigma).$$

Let E_1, E_2 and R be determined as before, and let $G_1 = E_1 \times R$, $G_2 = E_2 \times R$ be the induced representations of Σ. Then it is immediate that $G_1 \times G_2 = (E_1 \times R \times E_2) \times R$ is the self-representation of Σ induced by $E_1 \times R \times E_2$. It follows that

$$\Gamma_1(\Sigma) \times \Gamma_2(\Sigma) = (\Gamma_1 \times \Delta \times \Gamma_2)(\Sigma).$$

COROLLARY 1. *For any composites* Γ_1, Γ_2 *we have*

$$(\Gamma_1 \times \Gamma_2)(\Sigma) \leq \Gamma_1(\Sigma) \times \Gamma_2(\Sigma).$$

Since Δ contains the identity composite, $\Gamma_1 \times \Gamma_2 \leq \Gamma_1 \times \Delta \times \Gamma_2$. The corollary then follows from Theorem 3.

We also have the following partial converse of Theorem 2.

COROLLARY 2. *If* Γ_1 *and* Γ_2 *are simple composites such that*

$$\Delta \times \Gamma_1 \times \Delta = \Delta \times \Gamma_2 \times \Delta$$

then

$$\Gamma_1(\Sigma) = \Gamma_2(\Sigma).$$

For

$$(\Delta \times \Gamma_1 \times \Delta)(\Sigma) \leq \Delta(\Sigma) \times \Gamma_1(\Sigma) \times \Delta(\Sigma) = \Gamma_1(\Sigma).$$

Suppose now that $\Gamma = (K, S, T)$ is a simple non-singular composite of P. Then $(K : \mathrm{P}^S) = (K : \mathrm{P}^T)$ and so $(K : \Sigma^S) = (K : \Sigma^T)$. Since $\Sigma^S \Sigma^T$ is a subfield we have

$$(K:\Sigma^S) = (K:\Sigma^S\Sigma^T)(\Sigma^S\Sigma^T:\Sigma^S)$$
$$(K:\Sigma^T) = (K:\Sigma^S\Sigma^T)(\Sigma^S\Sigma^T:\Sigma^T).$$

Hence $(\Sigma^S\Sigma^T:\Sigma^S) = (\Sigma^S\Sigma^T:\Sigma^T)$. This proves that $\Gamma(\Sigma)$ is non-singular. It is evident from the definition that $\Gamma(\Sigma) = \Gamma^{-1}(\Sigma)$.

THEOREM 4. *If Γ is a non-singular simple composite, then the contraction $\Gamma(\Sigma)$ is non-singular and simple and $\Gamma(\Sigma)^{-1} = \Gamma^{-1}(\Sigma)$.*

4. The hypergroup of a separable field. We suppose now that P is finite and separable over a subfield Φ and we let $H_{P|\Phi}$ denote the hypergroup of simple composites of P over Φ (i. e., leaving the elements of Φ fixed). We recall that $H_{P|\Phi}$ is finite and the least common cover of all the Γ_i in $H_{P|\Phi}$ is the closed composite Γ of P over Φ. Any composite of P over Φ is the least common cover $\Sigma_k\Gamma_{i_k}$ of certain of the Γ_i in $H_{P|\Phi}$ and by the distributive law we have $(\Sigma\Gamma_{i_k}) \times (\Sigma\Gamma_{j_l}) = \Sigma\Gamma_{i_k} \times \Gamma_{j_l}$. If $\Gamma_1 \times \Gamma_2 = \Sigma\Gamma_{i_k}$, we define the set $\Gamma_{i_1}, \Gamma_{i_2}, \cdots$ to be the product $\Gamma_1\Gamma_2$ of the simple composites Γ_1 and Γ_2. This operation is the hypergroup operation in $H_{P|\Phi}$.

Let Σ be a field between P and Φ and let $H_{P|\Sigma}$ be the subhypergroup of $H_{P|\Phi}$ of simple composites of P over Σ. We recall that any subhypergroup of $H_{P|\Phi}$ is an $H_{P|\Sigma}$ and that this correspondence is $(1-1)$. If Δ is the closed composite corresponding to Σ, $\Delta = \Sigma\Delta_i$, Δ_i in $H_{P|\Sigma}$.

Now if $\Gamma_1 \epsilon H_{P|\Phi}$, the contraction $\Gamma'_1 = \Gamma_1(\Sigma)$ is in the hypergroup $H_{\Sigma|\Phi}$ of Σ over Φ. By Theorem 1 any Γ'_1 in $H_{\Sigma|\Phi}$ may be obtained in this way and by Corollary 1 $(\Gamma_1 \times \Gamma_2)' \leq \Gamma'_1 \times \Gamma'_2$. Thus the correspondence $\Gamma_1 \rightarrow \Gamma'_1$ is a homomorphism between the hypergroups $H_{P|\Phi}$ and $H_{\Sigma|\Phi}$. Evidently the kernel of this homomorphism is $H_{P|\Sigma}$. Now let Γ_1 and Γ_2 be two elements of $H_{P|\Phi}$, such that $\Gamma'_1 = \Gamma'_2$. Then by Theorem 2,

$$\Delta \times \Gamma_1 \times \Delta = \Delta \times \Gamma_2 \times \Delta.$$

Since $\Delta \times \Gamma_1 \times \Delta = \Sigma(\Delta_i \times \Gamma_1 \times \Delta_j)$ for Δ_i, Δ_j in $H_{P|\Sigma}$. $\Delta \times \Gamma_1 \times \Delta$ is the least common cover of all the elements in the double coset $H_{P|\Sigma}\Gamma_1 H_{P|\Sigma}$. Hence

$$\Delta \times \Gamma_1 \times \Delta = \Delta \times \Gamma_2 \times \Delta$$

implies that $H_{P|\Sigma}\Gamma_1 H_{P|\Sigma} = H_{P|\Sigma}\Gamma_2 H_{P|\Sigma}$. Next let $\Gamma_2 \epsilon H_{P|\Sigma}\Gamma_1 H_{P|\Sigma}$. Then $\Gamma_2 \leq \Delta \times \Gamma_1 \times \Delta$ and $\Gamma'_2 \leq \Gamma'_1$. Since Γ'_1 is simple, $\Gamma'_2 = \Gamma'_1$ and so $H_{P|\Sigma}\Gamma_1 H_{P|\Sigma} = H_{P|\Sigma}\Gamma_2 H_{P|\Sigma}$. This shows that two double cosets are either identical or their intersection is vacuous. The double cosets of $H_{P|\Sigma}$ give a division of $H_{P|\Phi}$ into mutually exclusive sets and those cosets define a factor hypergroup $H_{P|\Phi} /\!/ H_{P|\Sigma}$. It is clear also that the double cosets of

$H_{\Sigma|\Phi}$ are in $(1-1)$ correspondence with the elements of $H_{P|\Sigma}$, namely, each double coset is the inverse image of an element of $H_{\Sigma|\Phi}$ relative to the homomorphism between $H_{P|\Phi}$ and $H_{\Sigma|\Phi}$. We consider now the product $(H_{P|\Sigma}\Gamma_1 H_{P|\Sigma})(H_{P|\Sigma}\Gamma_2 H_{P|\Sigma})$. The least common cover of the elements of this set is the composite $\Delta \times \Gamma_1 \times \Delta \times \Gamma_2 \times \Delta$. By Theorem 3,

$$(\Delta \times \Gamma_1 \times \Delta \times \Gamma_2 \times \Delta)' = (\Delta \times \Gamma_1)' \times (\Delta \times \Gamma_2)'$$

and since $(\Delta \times \Gamma_1)' = \Gamma'_1$ and $(\Delta \times \Gamma_2)' = \Gamma'_2$ by the simplicity of Γ_1 and Γ_2, $(\Delta \times \Gamma_1 \times \Delta \times \Gamma_2 \times \Delta)' = \Gamma'_1 \times \Gamma'_2$. It follows that the double cosets contained in the product $(H_{P|\Sigma}\Gamma_1 H_{P|\Sigma})(H_{P|\Sigma}\Gamma_2 H_{P|\Sigma})$ are the double cosets corresponding to the elements of $\Gamma'_1\Gamma'_2$. Hence we have proved that $(H_{P|\Phi} /\!/ H_{P|\Sigma})$ is isomorphic to $H_{\Sigma|\Phi}$.

THEOREM 5. *Let* P *be finite and separable over* Φ *and let* Σ *be a field between* P *and* Φ. *Then the correspondence* $\Gamma_i \to \Gamma'_i = \Gamma_i(\Sigma)$ *is a homomorphism between the hypergroups* $H_{P|\Phi}$ *and* $H_{\Sigma|\Phi}$. *The kernel of this homomorphism is* $H_{P|\Sigma}$ *and the double cosets of* $H_{P|\Sigma}$ *form a hypergroup* $H_{P|\Phi} /\!/ H_{P|\Sigma}$ *isomorphic to* $H_{\Sigma|\Phi}$.

As has been shown by Dresher and Ore, $H_{P|\Phi} /\!/ H_{P|\Sigma}$ is a group if and only if $H_{P|\Sigma}$ is strongly normal in $H_{P|\Phi}$ in the sense that for any Γ_1, $\Gamma_1^{-1}H_{P|\Sigma}\Gamma_1 = H_{P|\Sigma}$. We recall also that $H_{\Sigma|\Phi}$ is a group if and only if Σ is normal over Φ. Hence we have

THEOREM 6. *Let* $P \geq \Sigma \geq \Phi$ *where* P *is separable and finite over* Φ. *Then* Σ *is normal over* Φ *if and only if* $H_{P|\Sigma}$ *is strongly normal in* $H_{P|\Phi}$. *When the condition is satisfied* $H_{P|\Phi} /\!/ H_{P|\Sigma}$ *is isomorphic to the Galois group* $H_{\Sigma|\Phi}$ *of* Σ *over* Φ.

Theorem 5 enables us to obtain very precise information on the nature of the hypergroup $H_{\Sigma|\Phi}$ of a finite separable extension Σ of Φ. For we may extend Σ to the finite separable and normal extension P over Φ. Then $H_{P|\Phi}$ is a group isomorphic to the Galois group of P over Σ and $H_{P|\Sigma}$ is the subgroup of $H_{P|\Phi}$ corresponding to the Galois group of P over Σ. Thus the hypergroup $H_{\Sigma|\Phi}$ is a hypergroup of double cosets of a finite group. Such hypergroups are known to have many important special properties. They are, for example, completely regular in the sense that for any Δ_1 in $H_{\Sigma|\Phi}$ the only solutions of either of the equations $\Delta_1\bar{\Delta} = \{1, \cdots\}$ or $\bar{\Delta}\Delta_1 = \{1, \cdots\}$ is $\bar{\Delta} = \Delta_1^{-1}$. In the remainder of this paper we shall give a direct proof of this property based on some general theorems on inverses of self-representation.

5. Properties of the inverse of a self-representation. Let E be a non-singular self-representation of P of rank n and let E^* be its inverse. Then if $\alpha^E = (\alpha E_{ij})$ and $\alpha^{E^*} = (\alpha E^*_{ij})$ we have the defining relations

$$\sum_k E_{ki}E^*_{jk} = \delta_{ij}, \qquad \sum E^*_{ki}E_{jk} = \delta_{ij}$$

where δ_{ij} is the 0 endomorphism or the identity according as $i \neq j$ or $i = j$. Suppose that $\mathfrak{R}$ is the double P-module and $x_1, \cdots, x_n$ a right P-basis giving rise to E, so that $\alpha x_i = \Sigma x_j(\alpha E_{ji})$. Then $x_1, \cdots, x_n$ is also a left P-basis. Similarly in the inverse double P-module $\mathfrak{R}^{-1}$ we have a basis $x^*_1, \cdots, x^*_n$ such that $\alpha x^*_i = \Sigma x^*_j(\alpha E^*_{ji})$. Finally we let $\mathfrak{R}'$ be the double P-module corresponding to the transposed representation E' and let $x'_1, \cdots, x'_n$ be a right P-basis such that $\alpha x'_i = \Sigma x'_j(\alpha E_{ij})$. Consider the product module $\mathfrak{P} = \mathfrak{R}' \times \mathfrak{R}^{-1}$. A right basis for this module is $x'_i x^*_j$, $i, j = 1, \cdots, n$. Thus the element $u = \Sigma x'_i x^*_i \neq 0$ in $\mathfrak{P}$. Moreover the following relations hold:

$$\alpha u = \alpha \Sigma x'_i x^*_i = \Sigma x'_j \alpha E_{ij} x^*_i = \Sigma x'_j x^*_k \alpha E_{ij} E^*_{ki} = (\Sigma x'_i x^*_i)\alpha = u\alpha.$$

Evidently the vector $u_\rho = \Sigma(x'_i\rho)x^*_i$ also satisfies the relation $\alpha u_\rho = u_\rho \alpha$. Suppose now that (K, T, S) is the composite of E^* and let $(K : P^S) = m$. Then there exist m elements $\rho_1, \cdots, \rho_m$ in P such that the matrices $\rho_1^{E^*}, \cdots, \rho_m^{E^*}$ are linearly independent over P. We assert that the vectors

$$u_{\rho_l} = \Sigma(x'_i\rho_l)x^*_i = \Sigma x'_i x^*_j(\rho_l E^*_{ji})$$

are linearly independent. For if $\Sigma u_{\rho_l}\zeta_l = 0$, $\Sigma(\rho_l E^*_{ji})\zeta_l = 0$ for all i, j and so $\Sigma \rho_l^{E^*}\zeta_l = 0$. Hence each $\zeta_l = 0$. If we choose a right basis of P to be $u_{\rho_1}, \cdots, u_{\rho_m}$ and $n^2 - m$ other vectors, the associated self-representation is

$$\begin{array}{cc} & \overbrace{\hphantom{\alpha \quad \ddots \quad \alpha}}^{m} \\ \alpha \rightarrow & \left(\begin{array}{c|c} \begin{matrix}\alpha & & \\ & \ddots & \\ & & \alpha\end{matrix} & * \\ \hline 0 & * \end{array}\right) \end{array}$$

and this representation is similar to $E' \times E^*$.

THEOREM 6. *Let E be a non-singular self-representation of a field* P, *E^* its inverse and E' its transposed. Then if $(K : \mathrm{P}^S) = m$ for the composite (K, T, S) of E^*, the identity self-representation occurs as an irreducible component of $E' \times E^*$ with a multiplicity $r \geq m$.*

We suppose now that P is finite and separable over Φ and let E be an irreducible self-representation of P over Φ (leaving the elements of Φ fixed). We know that the composite $\Gamma_1 = (K_1, S_1, T_1)$ associated with E is simple, and it follows from this that E is similar to the self-representation obtained by regarding K_1 as a double P-module. Hence rank $E = (K_1 : P^{T_1}) \equiv m$ and since Γ_1 is non-singular, $(K_1 : \mathrm{P}^{T_1}) = (K_1 : \mathrm{P}^{S_1})$.

We assume now that E is non-singular and let R be the regular representation of P over Φ. Consider the self-representation $R \times E$. Since the elements $\alpha R_{pq} \epsilon \Phi$, $\alpha^{R \times E} = \alpha^R \times 1_m$ so that $\alpha^{R \times E}$ is similar to the direct sum of R with itself taken m times. We recall that R is completely reducible, that any irreducible self-representation of P over Φ is similar to one of the components of R and that no two components of R are similar. It follows from this that the identity has the multiplicity m in $R \times E$. On the other hand, by Theorem 6, $(E^*)' \times E$ contains the identity with multiplicity $r \geq m$. Since $(E^*)'$ may be taken to be one of the irreducible components of R, we see that the multiplicity of the identity self-representation in $(E^*)' \times E$ is m and if G is any irreducible self-representation of P over Φ such that $G \times E$ contains the identity, then G is similar to $(E^*)'$ and hence to E^*.

Now let E be an arbitrary irreducible self-representation of P over Φ and let F be any irreducible self-representation of P over Φ with composite Γ_1^{-1}. Then by replacing E and F by similar self-representations, we see that $F \times E$ contains the identity as an m-fold component and if G is irreducible with composite $\neq \Gamma_1^{-1}$, $G \times E$ does not contain the identity as a component. By symmetry we see also that $E \times F$ contains the identity as an m-fold component and if G does not have the composite Γ_1^{-1}, $E \times G$ does not contain the identity. This proves the following theorem.

THEOREM 7. *Let* P *be finite and separable over Φ and let E be an irreducible self-representation of rank m of* P *over Φ. Then if F is an irreducible self-representation of* P *over Φ whose composite is the inverse of the composite of E, $E \times F$ and $F \times E$ contain the identity as an m-fold component. If G is an irreducible self-representation of* P *over Φ with composite different from the inverse of the composite of E, then $E \times G$ and $G \times E$ do not contain the identity as a component.*

THEOREM 8. *Let* P *be finite and separable over* Φ *and let* $\Gamma_1 \epsilon H_{\mathrm{P}|\Phi}$. *Then* $1 \epsilon \Gamma_1\Gamma_1^{-1}$ *and* $1 \epsilon \Gamma_1^{-1}\Gamma_1$ *and if* Γ_2 *is any element of* $H_{\mathrm{P}|\Phi} \neq \Gamma_1^{-1}$, *neither* $\Gamma_1\Gamma_2$ *nor* $\Gamma_2\Gamma_1$ *contain the identity.*

This result amounts to the statement that $H_{\mathrm{P}|\Phi}$ is a completely regular hypergroup.

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GALOIS THEORY OF PURELY INSEPARABLE FIELDS OF EXPONENT ONE.*

By N. Jacobson.

Let P be a purely inseparable extension of finite dimensionality of a field Φ. Then $P = \Phi(x_1, \cdots, x_m)$, where $x_i^p = \xi_i$ is in Φ and p is the characteristic of Φ. Without loss of generality we may assume that $(P : \Phi) = p^m$. Let $\mathfrak{D}(\Phi)$ denote the set of derivations of P over Φ.[1] Then $\mathfrak{D}$ is a vector space of dimensionality m over $\bar{P}$ the set of multiplications $\bar{\beta} : \xi \to \xi\beta$ in P and $\mathfrak{D}$ is closed under commutation and under the operation of taking p-th powers. We call any set $\mathfrak{E}$ of derivations in P having these closure properties a *restricted* P-*Lie* ring of derivations. We shall call $\mathfrak{E}$ *finite* if $(\mathfrak{E} : \bar{P})$ is finite. In a previous paper[2] we set up a $(1-1)$ correspondence between the fields Σ between P and Φ and the restricted P-Lie rings of derivations contained in $\mathfrak{D}$. In this note we obtain the following results: 1) the only elements of P that are constants relative to all the derivations in $\mathfrak{D}$ are the elements of Φ; and 2) if $\mathfrak{D}$ is any finite restricted P-Lie ring of derivations in P and Φ is the subfield of $\mathfrak{D}$-constants, then P is finite and purely inseparable of exponent 1 over Φ and $\mathfrak{D}$ is the complete set of derivations of P over Φ. Thus we have an order anti-isomorphism between the subfields Φ over which P is finite and purely inseparable of exponent 1 and the finite restricted P-Lie rings $\mathfrak{D}$ of derivations in P. This, of course, contains our earlier result. The improvement obtained here, analogous to that of Artin and Baer in the ordinary Galois theory, consists in showing that $\mathfrak{D}$ and Φ serve equally well as starting points of the Galois theory. We remark that the determination of the structure of finite restricted P-Lie rings of derivations is a consequence of our results. Our proofs are based on theorems on self-representations of fields recently obtained by the author.[3]

* Received April 27, 1944.

[1] For the definition of a derivation and the properties quoted in this paragraph, see the author's "Abstract derivation and Lie algebras," *Transactions of the American Mathematical Society*, vol. 42 (1937), pp. 206-224. This paper is referred to as D.

[2] D. p. 220.

[3] See "An extension of Galois theory to non-normal and non-separable fields," this *Journal*, vol. 66 (1944), pp. 1-29. We refer to this paper as E.

1. Let $P = \Phi(x_1, \cdots, x_m)$, $x_i^p = \xi_i$ in Φ and let $(P : \Phi) = p^m$ where p is the characteristic of Φ, and let $\mathfrak{D}(\Phi)$ be the restricted P-Lie ring of derivations of P over Φ. If $D \epsilon \mathfrak{D}$ the correspondence

$$\alpha \rightarrow \begin{pmatrix} \alpha & \alpha D \\ 0 & \alpha \end{pmatrix} \tag{1}$$

is a self-representation of P whose field of fixed elements contains Φ. We recall that there exists a derivation D such that the field of D-constants is Φ.[4] Let D have this property. Then the subfield of fixed elements under (1) is precisely Φ. Then by Theorem 13 of E.[3] any linear transformation in P over Φ is a polynomial in D with coefficients in $\bar{P}$. Consider the sequence $1, D, D^2, \cdots$ and let D^r be the first of these transformations that is right linearly dependent over $\bar{P}$ on $1, D, \cdots, D^{r-1}$. Then any polynomial in D and hence any linear transformation in P over Φ may be written in one and only one way in the form $\bar{\beta}_0 + D\bar{\beta}_1 + \cdots + D^{r-1}\bar{\beta}_{r-1}$. Thus if $\mathfrak{L}$ denotes the complete set of linear transformations of P over Φ, $(\mathfrak{L} : \bar{P}) = r$. Since $(P : \Phi) = p^m$ this gives $(\mathfrak{L} : \bar{\Phi}) = p^m r$. On the other hand, as is well known, $(P : \Phi) = p^m$ implies that $(\mathfrak{L} : \bar{\Phi}) = p^{2m}$. Hence $r = p^m$. Now the linear transformations

$$E = D\bar{\beta}_1 + D^p\bar{\beta}_p + D^{p^2}\bar{\beta}_{p^2} + \cdots + D^{p^{m-1}}\bar{\beta}_{p^{m-1}} \tag{2}$$

are derivations and since their totality is an m-dimensional space over $\bar{P}$, this totality coincides with $\mathfrak{D}$. We have therefore proved

LEMMA 1. *If D is a derivation in P over Φ such that the only D-constants are the elements in Φ then any derivation E in P over Φ is a p-polynomial* (2) *in D with coefficients* (*on the right*) *in* P.[5]

We shall require also the following

LEMMA 2. *If $\mathfrak{E}$ is a restricted P-Lie subring of $\mathfrak{D}$, there exists a derivation E in $\mathfrak{E}$ such that any derivation in $\mathfrak{D}$ is a p-polynomial over $\bar{P}$ in E.*[6]

As a consequence we have

LEMMA 3. *If $\mathfrak{E}$ is a restricted P-Lie subring of $\mathfrak{D}$ whose field of constants is Φ, then $\mathfrak{E} = \mathfrak{D}$.*

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[4] This result is due to Baer, "Algebraische Theorie der differentierbaren Funktionenkorper. I," *Sitzungsberichte Heidelberger Akad.*, 1927, pp. 15-32. Cf. also the author's paper "Classes of restricted Lie algebras of characteristic p, II," *Duke Mathematical Journal*, vol. 10 (1943), p. 111.

[5] This is proved in D. p. 218 by using the theory of linear differential equations.

[6] D. p. 219.

For the field of $\mathfrak{E}$-constants is evidently the same as the field of E-constants. Thus E is a derivation of P whose field of constants is Φ and by Lemma 1 any derivation in P over Φ is a p-polynomial in E.

2. We suppose now that P is any field of characteristic $p \neq 0$ and let $\mathfrak{D}$ be any finite restricted P-Lie ring of derivations in P. Let $D_1, \cdots, D_m$ be a (right) $\bar{\text{P}}$-basis of $\mathfrak{D}$ and let $\mathfrak{A}$ be the smallest ordinary ring of endomorphisms in P containing $\mathfrak{D}$ and $\bar{\text{P}}$. Then we have

LEMMA 4. *$\mathfrak{A}$ is finite dimensional over* $\bar{\text{P}}$.

Proof. Let $\mathfrak{A}'$ denote the totality of endomorphisms of the form $\Sigma D_1^{k_1} D_2^{k_2} \cdots D_m^{k_m} \bar{\beta}_{k_1 \ldots k_m}$ where $0 \leqq k_i < p$ and we set $D_i^0 = 1$. Evidently $\mathfrak{A} \geq \mathfrak{A}' \geq \mathfrak{D}, \text{P}$. Since D_i is a derivation for any β, $\bar{\beta} D_i = D_i \bar{\beta} + \overline{\beta D_i}$ and since $\mathfrak{D}$ is a restricted P-Lie ring, $D_i D_j = D_j D_i + \Sigma D_k \bar{\gamma}_{kij}$ and $D_i^p = \Sigma D_j \bar{\mu}_{ji}$. It is readily seen from these relations that $\mathfrak{A}'$ is a ring. Hence $\mathfrak{A}' = \mathfrak{A}$ and since any element in this ring is a linear combination of the p^m elements $D_1^{k_1} D_2^{k_2} \cdots D_m^{k_m}$, $(\mathfrak{A} : \bar{\text{P}}) \leq p^m$.

We may now prove the following

THEOREM. *Let $\mathfrak{D}$ be a finite restricted* P*-Lie ring of derivations in* P *and let Φ be the subfield of $\mathfrak{D}$-constants. Then* P *is finite and purely inseparable of exponent* 1 *over Φ and $\mathfrak{D}$ is the complete set of derivations of* P *over Φ.*

As above we let $D_1, \cdots, D_m$ be a $\bar{\text{P}}$-basis of $\mathfrak{D}$ and we form the self-representation E:

$$
(3) \qquad \alpha \to \begin{pmatrix} \alpha & \alpha D_1 & & & & & & \\ 0 & \alpha & & & & & & \\ & & \alpha & \alpha D_2 & & & & \\ & & 0 & \alpha & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \alpha & \alpha D_m \\ & & & & & & 0 & \alpha \end{pmatrix}.
$$

The field of fixed elements relative to this representation is Φ. By substituting for the elements β of the matrices in (3) the matrices β^E we obtain the self-representation $E \times E$. We write the resulting matrices as $\alpha^{E \times E} = (\alpha E^{(2)}{}_{kl})$ and note that the endomorphisms $E^{(2)}{}_{kl}$ include $1, D_i, D_i D_j, i, j = 1, \cdots, m$. Similarly we form the product $E \times E \times E$ and obtain $\alpha^{E \times E \times E} = (\alpha E^{(3)}{}_{pq})$

where the $E^{(3)}{}_{pq}$ include $1, D_i, D_iD_j, D_iD_jD_k$. Continuing in this way we obtain finally a self-representation $F = E \times \cdots \times E$ whose endomorphisms include all the elements $D_1^{k_1} \cdots D_m^{k_m}$, $0 \leq k_i < p$. Since these elements generate the ring $\mathfrak{A}$ it follows that the composite associated with F is closed and hence P is finite over the field of fixed elements under F.[7] Since this field is the same as the field Φ of fixed elements under E, $(\mathrm{P} : \Phi)$ is finite. Now let x be any element of p and let $D \in \mathfrak{D}$. Then $x^pD = 0$. Hence $x^p = \xi$ is in Φ and P has exponent 1 over Φ. This proves the first part of the theorem. The second part of the theorem is an immediate consequence of Lemma 3.

We remark that by the above theorem and a previous result[8] we obtain the

COROLLARY. *Let $\mathfrak{D}$ be a finite restricted* P*-Lie ring of derivations in* P *and let Φ be the subfield of $\mathfrak{D}$-constants. Then unless* $\mathrm{P} = \Phi(x)$, $x^2 = \xi$ *in* Φ, *$\mathfrak{D}$ is a simple Lie algebra over Φ.*

Now by Lemma 1, if $\mathrm{P} = \Phi(x_1, \cdots, x_m)$, $x_i^p = \xi_i$ in Φ and $\mathfrak{D}$ is the derivation ring of P over Φ, then Φ is the field of $\mathfrak{D}$-constants. This completes the proof of the $(1 - 1)$ correspondence between the subfields Φ of P over which P is finite and purely inseparable of exponent 1 and the finite restricted P-Lie rings $\mathfrak{D}(\Phi)$ in P. Evidently if $\mathrm{P} \geq \Sigma \geq \Phi$ then $\mathfrak{D}(\Sigma) \leq \mathfrak{D}(\Phi)$ and conversely.

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[7] E. p. 19.

[8] D. p. 218.

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